MODELS FOR COMPUTER - AIDED DESIGN

(Sensitivity and Optimization Models)

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by
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CERTIFICATE

Certified that this work has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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LIST OF SYMBOLS

Symbol	Description and Remarks (if any)
(a) Scalars	
α	
β	
γ	
δ	
C	
G	
i	
j	
k	
L	
m	
n	
q	
R	
S	
T	
v	
w	used for frequency.
w_1	
w_2	

Symbol Description and Remarks

(b) Matrices (including column matrices)

A

A_{12}

B

B_{11}

C

D_d^s

D_r^s

G_4

I

L

P

P_1

Q

R_3

U

V

X

Y

δP

Δ

Symbol Description and Remarks

(c) Functions

D_d D_r	Vector-Valued functions
G	
I	
R	
S	
T	
V	

Remark : Some symbols have been given more than one meaning. Such a situation could not be avoided because of the restricted set of symbols available on the typewriter. The intended meaning of a symbol is always clear from the context in which it is used.

S Y N O P S I S

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Models for Computer-aided Design
(Sensitivity and Optimization Models)

There is increasing interest of late in extending the possibilities of computer-aided design for the solution of a large class of problems. A large amount of the present work in this area has however been confined to network design problems. An excellent cross-section of such work is found in the special issue of the Proceedings of IEEE on computer-aided design.

Intimately associated with this problem of computer-aided design are the relative merits of on-line computation versus batch processing. In either of these, the theoretical studies in developing proper simulation models have engaged the attention of many research workers.

One important theoretical study of this nature is sensitivity analysis which constitutes an important link between analysis and design. In this thesis, sensitivity models have been developed which are algorithmic in nature and hence can easily be incorporated into computer formulation and solution.

These computer models for sensitivity analysis constitute an extension of the state-space formulation of linear and non-linear networks. While reviewing this necessary material, a contribution is also made for a rigorous inclusion of non-linear

components which have non-monotonic characteristics. The equations for such components do not satisfy an essential condition to satisfy the sufficiency conditions for the existence of solutions to a system of non-linear algebraic equations. This difficulty is circumvented by considering such components as dependent drivers. Thus, restrictions in the form of terminal equations are shifted to restrictions in topological considerations. These latter conditions are stated.

If design concepts are to be based on this analysis, further study of the models is necessary. Thus, for example, higher-order sensitivity coefficients are required for the formulation of the minimization of sensitivity of a network. Models for determining higher-order sensitivity coefficients have also been developed.

For establishing design objectives based on sensitivity, we need measures of sensitivity. These measures are defined both in the frequency-domain as well as in the time-domain. The models that are developed here are equally applicable for either frequency- or time-domain analyses and therefore lend themselves for design purposes based on sensitivity measures.

In stability analyses, the question of sensitivity appears quite naturally. For example, Lyapunov defines local stability in terms of the boundedness of the sensitivity of the response to changes in initial conditions. These aspects have been discussed here quite explicitly.

Determination of nth order sensitivity coefficients for a linear network depends on the response of the given network and a knowledge

of the sensitivity coefficients of order (n-1). However, the basic data that is required for determining higher order sensitivity coefficients is just the information about the network graph and its component terminal equations; the rest of the information required for the final solution can be determined from these. For a non-linear network, however, additional information about the component terminal equations is necessary for every increase in the order of the sensitivity coefficient.

The next important problem considered in this thesis is the optimization problem. Again, the thesis is that it would be advantageous to develop network models for this study so as to render the formulation of the optimization problem condueive to digital computer use. A proper network interpretation has been given to the co-state equations of a network, thus preserving the structural features of the problem, which are usually not considered in control theory.

Having systematized the formulation of co-state equations in terms of network models, it is shown that in certain studies of the optimization problem, the existing network analysis programmes can be used for the solution of the resulting two-point boundary value problem. It is also shown that sufficient conditions for optimality, such as the ones discussed by Robbins, do not admit a network interpretation.

Another type of optimization study which has interested research workers is the parameter optimization problem. A component-oriented formulation of this problem is presented.

The component-oriented formulation has distinct advantages over other formulation procedures where the parameters do not correspond to the individual components. In the procedure given here, the need for algebraic manipulation is minimized, and error analysis becomes quite straight-forward.

One of the interesting theoretical questions in parameter-optimization studies concerns the search procedures within the parameter space; it would be interesting to know that part of the parameter space in which the search procedures are effective. This question has been partially answered in the thesis.

Leeds and Ugron have conjectured that if in a network designed for minimal sensitivity the number of components is increased it is always possible to reduce the sensitivity. This conjecture is shown to be incorrect by means of a counter-example.

This thesis uses network language exclusively. However, the analysis is equally applicable to all discrete physical systems. The range of applicability is achieved by means of graph-theoretic formulation procedures which are quite valid for other systems also. Exclusive use of the network language is made merely for the sake of simplicity in presentation.

1. Introduction

There is increasing interest of late in extending the possibilities of computer-aided design for the solution of a large class of problems. A large amount of the present work in this area has however been confined to network design problems. An excellent cross-section of such work is found in (16). Other examples of interest in this field are (2,14,19).

Initimately associated with this problem of computer-aided design are the relative merits of on-line computation versus batch processing. In either of these, the theoretical studies in developing proper simulation models have engaged the attention of many research workers (7,9,12).

One important theoretical study of this nature is sensitivity analysis which constitutes an important link between analysis and design. Early workers in this area (for example, (17)) have mostly developed models which are analytical in nature, but which are not necessarily amenable for transformation into digital simulation models.

In this thesis, sensitivity models have been developed which are algorithmic in nature and hence can easily be incorporated into computer formulation and solution. The general nature of these analyses is quite similar to the underlying procedures of problem-oriented programmes such as (7,13) so that the latter's range of usefulness could be enhanced when these additional features are incorporated.

When network analysis programmes are not endowed with the ability for sensitivity analyses, several simulations of the same network configuration with slight changes in component values are necessary before one could arrive at an understanding about the sensitivity of the network response with respect to component values. Network programmes which incorporate the sensitivity models obviate the necessity for several simulations of the same network.

We have so far dealt with analytical aspects of sensitivity. However, if design concepts are to be based on this analysis, further study of the models is necessary. Thus, for example, higher-order sensitivity coefficients are required for the formulation of the minimization of sensitivity of a network as discussed in chapter 4. Models for determining higher order sensitivity coefficients have also been developed. These models also constitute a generalization of the procedure for determining first-order sensitivity coefficients.

For establishing design objectives based on sensitivity, we need measures of sensitivity. These can be defined either in the frequency domain or in the time-domain. The models that are developed here are equally applicable for either frequency-or time-domain analyses and therefore lend themselves for design purposes based on sensitivity measures. Chapter 3 deals with these questions.

In stability analyses, the question of sensitivity appears quite naturally. For example, Lyapunov defines stability in terms of the boundedness of the sensitivity of the response to changes in initial conditions. These aspects have been discussed quite explicitly in chapter 3.

In chapter 4, the optimization problem is discussed. Again, the thesis is that it would be advantageous to develop network models for this study so as to render the formulation of the optimization problem conducive to digital computer use. In some restricted problems, it is possible to use the initial-value programmes directly for the solution of the optimization problem. In some other cases, it may be possible to solve the two-point boundary-value problem of optimization by repeatedly carrying out initial-value analyses. With this motivation, a proper network interpretation has been given to the co-state equations of a network, thus preserving the structural features of the problem, which are usually not considered in control theory (11,18). This additional feature can also be incorporated into the existing network analysis programmes.

Another type of optimization study which has interested research workers is the parameter optimization problem (1,14). A component-level formulation of this problem is presented in chapter 4. There is scope for computerization of this formulation. The component-level formulation has distinct advantages over other formulation procedures using indirect parameters which do not correspond to the individual components. The illustrative example presented shows how a minimum sensitivity network can be obtained by this formulation.

This thesis has used network language exclusively. However, since the analysis is throughout based on graph theoretic concepts, it is equally applicable to all discrete physical systems. Exclusive use of the network language is done merely for the sake of simplicity in presentation.

2. Computer Formulation and Solution of Linear and Nonlinear Networks.

2.1 Introduction : The development of the main body of this thesis, viz. sensitivity analyses and optimization problems, presupposes a knowledge of time-domain analysis of linear and nonlinear networks. Furthermore, since in these inquiries structural aspects are preserved, state models based on graph-theoretic aspects are found essential. Consequently, these latter models are briefly reviewed in this chapter.

2.2 Linear Graphs : This section is devoted to some definitions and theorems of graph theory. The theorems are stated here without proof. Their proofs can be found in (15).

Definition 2.2.1 Edge : A directed line segment whose ends are distinct is called an edge. An edge may meet other edges at either end, but may not intersect, or be intersected by, other edges.

Definition 2.2.2 Vertex : A common end-point of any number of edges is called a vertex. The number of edges meeting at a vertex is called the degree of the vertex. Trivially, an isolated point (which is the end point of no, or zero, edge) can be considered a vertex of degree zero.

Definition 2.2.3 Graphs and Subgraphs : A collection of edges and vertices such that no two edges have a point in common which is not a vertex is called a graph. A subcollection of the edges and vertices of a graph is called a subgraph.

Definition 2.2.4 Path : A sequence of edges which satisfies the following conditions is called a path.

(i) Each edge in the sequence except the first has a vertex in common with the edge which precedes it in the sequence, and each edge except the last has a vertex in common with the edge which follows it in the sequence.

(ii) No edge or vertex appears repeatedly in the sequence.

Definition 2.2.5 Initial, final, and terminal vertices of a path :

That vertex of the first edge which is not common to the second edge is called its initial vertex. Similarly, that vertex of the last edge which is not common to the previous edge is called its final vertex. The initial and final vertices of a path are collectively called its terminal vertices.

Definition 2.2.6 Circuit : If the initial vertex of a path coincides with its final vertex the path is called a circuit.

Definition 2.2.7 Connected Graph : A graph is said to be connected if there exists a path between any two of its vertices.

A graph G which has n connected subgraphs such that

(a) no two of them have a vertex in common, and

(b) their union is the graph G,

is said to be in n parts. Those subgraphs are said to be parts of G. A connected graph is clearly in one part.

Definition 2.2.8 Cutset : For a connected graph G, a subgraph C which has the properties :

(a) removal of C from G will leave G in two parts,

(b) no proper subset of the edges in C has the property (a),

is called a cutset.

Definition 2.2.9 Tree : A connected subgraph of a connected graph G containing all its vertices and no circuits is called a tree of G.

In a connected graph G of v vertices any tree contains exactly (v-1) edges. The edges of a tree are called its <u>branches</u> while the remaining edges of G are called <u>chords</u>. The collection of all chords is called the <u>complement</u> of the tree, or its <u>co-tree</u>.

Definition 2.2.10 Fundamental circuit (f-circuit) and fundamental cutset (f-cutset).

In connection with a connected graph G and its tree T, a circuit containing exactly one chord is called a f-circuit. Each chord belongs to a unique f-circuit. The number of f-circuits is therefore (e - v + 1) where e is the number of edges in G.

A cutset containing exactly one branch is called an f-cutset. Again, each branch belongs to a unique f-cutset, and hence the number of f-cutsets is (v - 1).

Definition 2.2.11 : f-cutset and f-circuit matrices : Given a connected graph G and a tree T, we now define two matrices A and B, called the f-cutset matrix and the f-circuit matrix respectively.

The rows of A correspond to f-cutsets defined by the branches and its columns correspond to all edges. The typical element a_{ij} is defined as follows :

a_{ij} = +1 if the jth edge is in the f-cutset of the ith branch and their orientations coincide,

= -1 if the jth edge is in the f-cutset of the ith branch and their orientations are opposite,

if the jth edge is note in the f-cutset of the ith branch.

When the columns of A corresponding to the tree are placed first and when the order of these columns coincides with the row order, A has the form :

$$A = \begin{bmatrix} U & A_{12} \end{bmatrix} \qquad 2.2.1$$

where U is a unit matrix of the appropriate size*.

The rows of B correspond to f-circuits defined by chords and its columns correspond to all edges. The typical element b_{ij} is defined as follows :

$b_{ij} = +1$ if the jth edge is in the f-circuit of the ith chord and their orientations coincide,

$= -1$ if the jth edge is in the f-circuit of the ith chord and their orientations are opposite,

$= 0$ if the jth edge is not in the f-circuit of the ith chord.

When the columns of B corresponding to the co-tree are placed last and when the order of these columns coincides with the row order, B has the form :

$$B = \begin{bmatrix} B_{11} & U \end{bmatrix} \quad . \qquad 2.2.2$$

When the columns of the matrices A and B are matrices satisfy the orthogonality condition

*The symbol U will always be used in this sense.

$$BA' = \underline{0} \qquad 2.2.3$$

where the prime denotes transpose ; and, consequently, if A and B are in the form of Eqs. 2.2.1 and 2.2.2 we have

$$B_{11} = -A'_{12} \qquad 2.2.4$$

2.3 Graphs of Networks :

A directed graph can be associated with an electrical network with lumped parameters. In this section we shall discuss this correlation. A detailed discussion can be found in (5).

Each edge in a network graph corresponds to a pair of voltage and current measurements between two terminals of some component (see Figures 2.3.1 and 2.3.2). A two-terminal component is represented by a single edge, and in general an n-terminal component is represented by a tree (also called a "terminal graph") in the n-terminal complete graph.

The cutset and circuit postulates governing the currents and voltages of the network are :

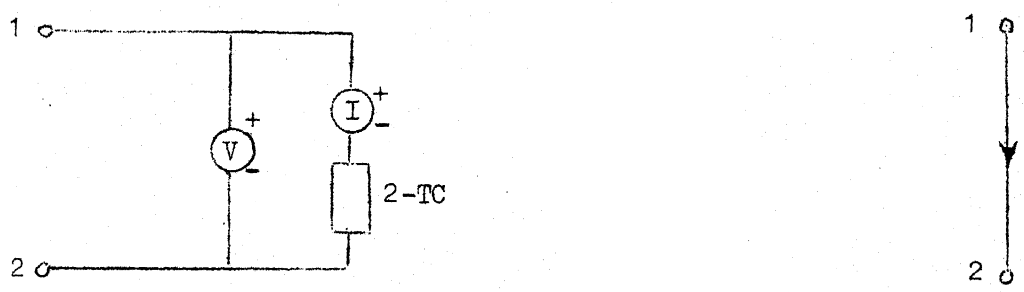
$$A\, I(t) = \underline{0} \qquad 2.3.1$$

and

$$B\, V(t) = \underline{0} \qquad 2.3.2$$

where the matrices A and B refer to the network graph, and I and V are vectors whose elements are the current and voltage variables associated with the edges of the network graph*, arranged in the

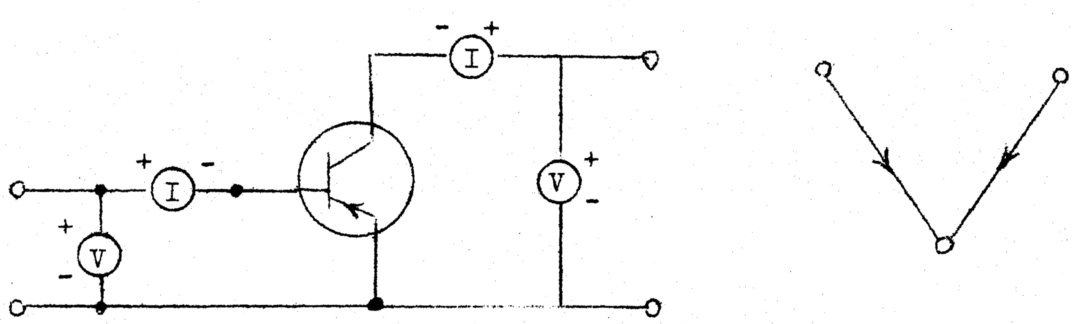
*For simplicity of expression, these current and voltage variables will hencefourth be called the currents and voltages of the graph. The distinction between the mathematical model and its physical counterpart will not be blurred by such references made in the proper context.



(a)
Direction-convention for voltage and current measurement

Direction-convention for corresponding edge in the network graph.

Fig. 2.3.1 Correlation between component measurement and network graph.



(a)
Sufficient measurements for measuring transistor characteristics

(b)
Graph of a transistor

Fig. 2.3.2 Graph of a multiterminal component.

same order as the columns of A and B.

Equations 2.3.1 and 2.3.2 together provide e independent relations in 2e variables; an equal number of independent relations in the same 2e variables is contributed by the component characteristics discussed in the following sections.

2.4 Component Characteristics :

First we shall consider a restricted class of components, namely, linear components. By describing a class of components we would also have described a class of networks - networks containing only components of the described class. Next, we shall move on to a more general set of nonlinear characteristics. These are defined in such a manner that the linear characteristics become a special case of the corresponding nonlinear ones. A network of nonlinear components may therefore in general contain linear components also.

The solution of linear networks in both time-and frequency-domains is already textbook material (see, for example, (5)). A method of analysis for networks of nonlinear two-terminal components was given in (13).

Furthermore, the formulation of networks containing multi-terminal algebraic components is also included here. The characteristics of such components are obtained as an extension of the characteristics of two-terminal resistors. However, it is sometimes more convenient to replace multiterminal algebraic components by their equivalent networks of two terminal components for the purposes of analysis. In fact, such representations are made use of

in computer studies. We shall make use of both these representations in our further study .

2.5 Linear Components :

Four types of two-terminal components and one type of multiterminal component come under this classification. They are:

(a) Two-terminal resistors.

The terminal equations can be written either in the form

$$v(t) = R\, i(t) \qquad 2.5.1$$

or in the form

$$i(t) = G\, v(t) \qquad 2.5.2$$

(b) Inductors.

The general form of their terminal equations is

$$v(t) = L \frac{di}{dt} \qquad 2.5.3$$

(c) Capacitors.

The general form of their terminal equations is

$$i(t) = C\frac{dv}{dt} \qquad 2.5.4$$

In the above equations, the parameters R,G, L and C may be positive constants or positive, bounded functions of time. A component whose parameter is constant is said to be time-invariant.

(d) Independent drivers.

The voltage or the current of such a component is a specified function of time. The general forms of their terminal equations are :

$$i = i(t) \qquad 2.5.5$$

or

$$v = v(t) \qquad 2.5.6$$

A driver whose voltage is specified is called a voltage driver, and a driver whose current is specified is called a current driver. By definition, independent drivers are considered to be time-invariant components.

(e) n-terminal resistors.

Such a component can be represented by a tree graph of $(n - 1)$ edges and the terminal equations

$$X(t) = D\,Y(t) \qquad 2.5.7$$

corresponding to these edges.

The notations $X(t)$ and $Y(t)$ for the column matrices (instead of $V(t)$ and $I(t)$) are used to allow for the possibility for the voltages and currents appearing in a mixed form.

The vector $X(t)$ in Eq. 2.5.7 consists of voltages of some (possibly none or all) of the edges and the currents of the remaining edges; and $Y(t)$ is the vector of the currents of those edges whose voltages are in $X(t)$, and the voltages of those whose currents are in $X(t)$. Such vectors $X(t)$ and $Y(t)$ are said to be <u>complementary</u>.

In Eq. 2.5.7, D is a matrix whose elements may be either constants or functions of time. If all the elements of D are constants, then the component is said to be time-invariant.

A network of only linear components is said to be linear. In addition, if all its components are time-invariant then the network is said to be linear time-invariant. A network that is not time-invariant is said to be time-varying.

In another representation of a multiterminal component we replace it for the purposes of analysis, by a network of two-terminal resistors and dependent drivers. Only two types of dependent drivers are necessary to be able to represent any multiterminal component of the type described by Eq. 2.5.7. In addition to these two more types of dependent drivers are defined for convenience. These four types are given below with their characteristics.

Type 1

$$v_2(t) = \alpha\, v_1(t) \qquad 2.5.8$$

Type 2

$$v_2(t) = \beta\, i_1(t) \qquad 2.5.9$$

Type 3

$$i_2(t) = \gamma\, v_1(t) \qquad 2.5.10$$

Type 4

$$i_2(t) = \delta\, i_1(t) \qquad 2.5.11$$

Only types 2 and 3 are essential, since types 1 and 4 can be derived from them.

Example 2.5.1

The multiterminal component represented by the graph of Figure 2.5.1(a) and the terminal equations

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} (t) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} (t) \qquad 2.5.12$$

can be replaced by the network of Figure 2.5.1(b) using dependent drivers of types 1 and 4. Alternatively; it can also be represented by the network of Figure 2.5.1(c) when only dependent drivers of types 2 and 3 are used.

2.6 Nonlinear Networks

The generalized terminal equations for nonlinear component characteristics are given below. Independent drivers, which are not mentioned below, remain unchanged.

(a) Two-terminal resistors

$$v(t) = R(i(t);t) \qquad 2.6.1(a)$$

or

$$i(t) = G(v(t);t) \qquad 2.6.1(b)$$

(b) Inductors

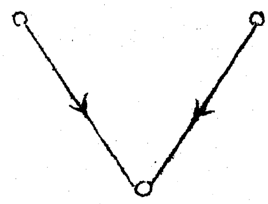
$$v(t) = \frac{d\phi}{dt}$$

$$i(t) = T(\phi(t);\ t) \qquad 2.6.2$$

(c) Capacitors

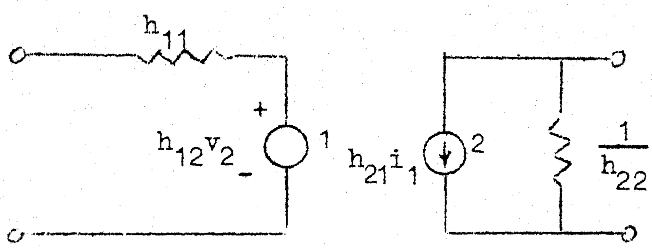
$$i(t) = \frac{dq}{dt}$$

$$v(t) = S(q(t);t) \qquad 2.6.3$$



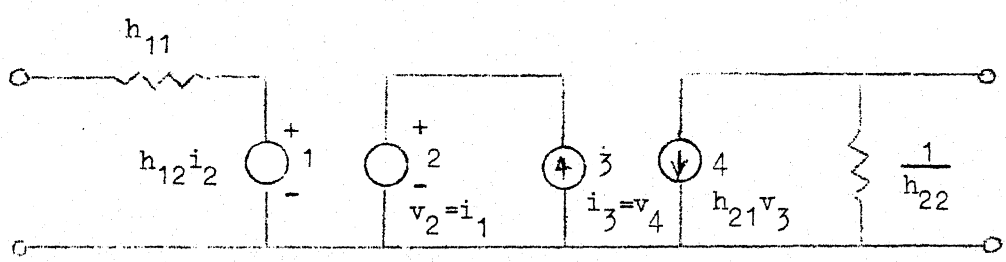
(a)

Graph of the component in Example2.5.1



(b)

Network using dependent drivers of types 1 and 4.



(c)

Network using dependent drivers of types 2 and 3.

Fig. 2.5.1 Example of the dual representation of a multiterminal component.

(e) Multiterminal resistors

$$X(t) = D(Y(t); t) \qquad 2.6.4$$

Here, again, dependent drivers may be introduced to replace multiterminal resistors. The four-way classification of dependent drivers used in the linear case will be followed here. These non-linear characteristics are represented by the equations :

Type 1

$$v_2(t) = V(v_1(t); t) \qquad 2.6.5$$

Type 2

$$v_2(t) = V(i_1(t); t) \qquad 2.6.6$$

Type 3

$$i_2(t) = I(v_1(t); t) \qquad 2.6.7$$

Type 4

$$i_2(t) = I(i_1(t); t) \qquad 2.6.8$$

When the function (R, T,S,D,V, or I) in the characteristics of a component does not depend on time, the component is said to be time-invariant. When a network contains only time-invariant components, the network is said to be time-invariant.

2.7 Overspecification, Degeneracy, and Restrictions of Structure.

The existence of a circuit of voltage drivers or a cutset of current drivers amounts to overspecification. The existence of a circuit of voltage drivers and capacitors or a cutset of current

drivers and inductors is termed as a degeneracy. A degeneracy may give rise to infinite currents or voltages in the solution of the network. The following analysis presumes the absence of over-specifications and degeneracies. For such networks, the network graph will contain a tree which includes all the voltage drivers and capacitors and excludes all the current drivers and inductors. Such a tree is said to be maximal.

The A and B matrices defined for a maximal tree are used for the formulation. This tree is called the formulation tree. The voltages of the branches and the currents of the chords of the formulation tree are called the primary variables while the remaining voltages and currents are called the secondary variables. The secondary variables can be expressed in terms of the primary variables using Eqs. 2.3.1 and 2.3.2 as follows :

$$V_2(t) = -B_{11}\, V_1(t) \qquad 2.7.1$$

and

$$I_1(t) = -A_{12}\, I_2(t) \qquad 2.7.2$$

where the subscripts 1 and 2 of V and I denote the tree- and co-tree - variables respectively.

For the proofs of the results stated in this section see (5).

2.8 Network equations for linear networks.

When the restrictions of the last section are imposed on the network graph, we have six classifications of edges :

1. Branches corresponding to voltage drivers (dependent as well as independent drivers).
2. Branches corresponding to capacitors.
3. Branches corresponding to resistors.
4. Chords corresponding to resistors.
5. Chords corresponding to inductors.
6. Chords corresponding to current drivers (dependent as well as independent drivers).

On the basis of the six-way classification, the V and I vectors and the premultiplying A and B matrices are partitioned similarly. Correspondingly, we rewrite the f-cutset and f-circuit equations :

$$\left[\begin{array}{c|ccc} & A_{11} & A_{12} & A_{13} \\ U & A_{21} & A_{22} & A_{23} \\ & A_{31} & A_{32} & A_{33} \end{array}\right] \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} (t) = \underline{0} \qquad 2.8.1$$

and

$$\left[\begin{array}{ccc|c} B_{11} & B_{12} & B_{13} & \\ B_{21} & B_{22} & B_{23} & U \\ B_{31} & B_{32} & B_{33} & \end{array}\right] \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} (t) = \underline{0} \qquad 2.8.2$$

The terminal equations of capacitors and inductors can be grouped as follows :

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_5 \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} I_2 \\ V_5 \end{bmatrix} (t) \qquad 2.8.3$$

Using Eqs. 2.8.1 and 2.8.2 we can rewrite Eq. 2.8.3 in terms of primary variables as follows :

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_5 \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_5 \end{bmatrix} (t) + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \begin{bmatrix} V_3 \\ I_4 \end{bmatrix} (t) \right.$$

$$\left. + \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ I_6 \end{bmatrix} (t) \right\} \qquad 2.8.4$$

Among the variables on the right of Eq. 2.8.4, V_2 and I_5 together constitute the state vector of the network, namely, the capacitor voltages and inductor currents.

The solution of Eq. 2.8.4 depends on the determination of the variables in the vectors $\begin{bmatrix} V_1 \\ I_6 \end{bmatrix}$ and $\begin{bmatrix} V_3 \\ I_4 \end{bmatrix}$. We shall discuss the determination of these variables next.

The elements of the vector $\begin{bmatrix} V_1 \\ I_6 \end{bmatrix}$ are the primary variables of drivers. If all the drivers in the network are independent, these variables can be determined as functions of time. When dependent drivers are present, however, the values of the variables

in the vector $\begin{bmatrix} V_1 \\ \\ I_6 \end{bmatrix}$ can be found if :

(a) the dependent drivers depend only on the primary variables of other drivers, capacitors, and inductors; and

(b) it is possible to arrange the dependent drivers in a sequence such that no dependent driver depends on any driver which follows it in the sequence.

If these conditions are satisfied, the primary variables of the dependent drivers can be determined in the order of any sequence which satisfies condition (b) above.

Example 2.8.1

Figures 2.8.1(a), 2.8.1(b) show networks in which the conditions (a) and (b), respectively, are violated. Figure 2.8.1(c) shows a network in which these conditions are satisfied. Condition (b) is satisfied by the sequence (1,2). Once the voltages of the capacitors are known, the voltage of driver 1 and the current of driver 2 can be found in that order; since we can write

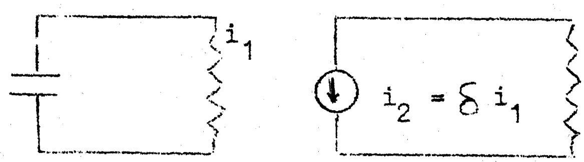
$$v_3 = \alpha v_1 ,$$

and

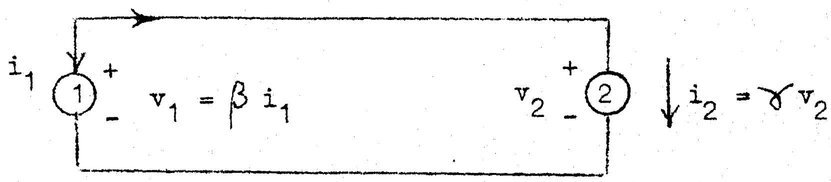
$$i_5 = \gamma (v_3 - v_4)$$

v_1, v_3 and v_4 are all known at the time when they are needed.

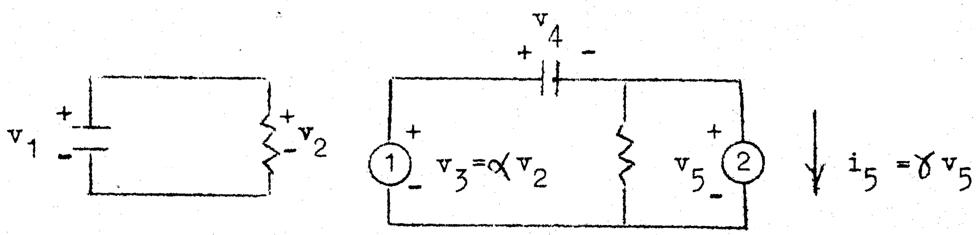
Next we shall consider two procedures for the determination of the elements of the vector $\begin{bmatrix} V_3 \\ I_4 \end{bmatrix}$.



(a)



(b)



(c)

Fig. 2.8.1 Networks for demonstrating the determination of the primary variables of dependent drivers.

The resistor equations can be handled by writing their characteristics explicitly in the primary variables :

$$\begin{bmatrix} V_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} I_3 \\ V_4 \end{bmatrix} \qquad 2.8.5$$

Substituting the values of I_3 and V_4 in terms of primary variables and transposing the term containing $\begin{bmatrix} V_3 \\ I_4 \end{bmatrix}$ to the left, we get

$$\left\{ U + \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \right\} \begin{bmatrix} V_3 \\ I_4 \end{bmatrix}$$

$$= \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{32} \\ -B_{12} & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_5 \end{bmatrix} + \begin{bmatrix} 0 & -A_{33} \\ -B_{11} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ I_6 \end{bmatrix} \right\}$$

$$- \quad 2.8.6$$

It has been proved in (5) that the coefficient matrix on the left of Eq. 2.8.6 is nonsingular. Premultiplying Eq. 2.8.6 by the inverse of that matrix and substituting the resulting value of $\begin{bmatrix} V_3 \\ I_4 \end{bmatrix}$ into Eq. 2.8.4 we get the dynamical equations (state equations) for the network :

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_5 \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} M \begin{bmatrix} 0 & -A_{32} \\ -B_{12} & 0 \end{bmatrix} \right\} \begin{bmatrix} V_2 \\ I_5 \end{bmatrix}$$

$$+ \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} M \begin{bmatrix} 0 & -A_{33} \\ B_{11} & 0 \end{bmatrix} \right\} \begin{bmatrix} V_1 \\ I_6 \end{bmatrix}$$

$$- \quad 2.8.7$$

where

$$M = \left\{ U + \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \right\}^{-1} \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \qquad 2.8.8$$

When dealing with nonlinear networks, the above formulation of resistor variables (Eqs. 2.8.5 to 2.8.8) is not possible. A formulation similar to the nonlinear case is given below :

First, the characteristics are written explicitly in the currents as follows :

$$\begin{bmatrix} I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} G_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} V_3 \\ V_4 \end{bmatrix} \qquad 2.8.9$$

A part of the f-cutset equations (Eq. 2.3.1) can be re-written making use of Eq. 2.8.9 as :

$$\begin{bmatrix} U & A_{31} \end{bmatrix} \begin{bmatrix} G_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} V_3 \\ V_4 \end{bmatrix} + A_{32}\, I_5 + A_{33}\, I_6 = \underline{0} \qquad 2.8.10$$

Substituting the values of the secondary voltages V_4 from the f-circuit equations (Eq. 2.3.2) we get

$$\begin{bmatrix} U & A_{31} \end{bmatrix} \begin{bmatrix} G_3 & 0 \\ 0 & G_4 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ -B_{11} \end{bmatrix} V_1 + \begin{bmatrix} 0 \\ -B_{12} \end{bmatrix} V_2 + \begin{bmatrix} U \\ -B_{13} \end{bmatrix} V_3 \right\}$$

$$+ A_{32} I_5 + A_{33} I_6 = \underline{0} \qquad 2.8.11$$

Once the solution to Eq. 2.8.11 is determined, we can solve the network by integrating Eq. 2.8.4.

We can treat Eq. 2.8.11 as a special case of the nonlinear resistor equations (Eq. 2.9.3) of the following section, and use the algorithm described there to obtain its solutions.

2.9 Extension of Network Equations to Nonlinear Networks.

The general nature of the formulation is the same as in the linear case. The equations of capacitors and inductors, having the forms of Eqs. 2.6.2 and 2.6.3 can be grouped together as follows :

$$\frac{d}{dt} \begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} I_2 \\ V_5 \end{bmatrix} \qquad 2.9.1$$

Substituting the value of the right-hand member of Eq. 2.9.1 in terms of primary variables we get :

$$\frac{d}{dt}\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_5 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \begin{bmatrix} V_3 \\ I_4 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ I_6 \end{bmatrix} \qquad 2.9.2$$

The elements of $\begin{bmatrix} V_2 \\ I_5 \end{bmatrix}$ can be found by the nonlinear part of the capacitor and inductor characteristics, having the forms of Eqs. 2.6.2 and 2.6.3, which can be grouped together as follows :

$$\begin{bmatrix} V_2 \\ I_5 \end{bmatrix} = D_d \left(\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix} \right) \qquad 2.9.3$$

It may be noted that the ith component of the function D_d depends only on the ith element of $\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}$.

The conditions on dependent drivers remain the same as in the previous section, and their characteristics can be simply used for obtaining the elements of $\begin{bmatrix} V_1 \\ I_6 \end{bmatrix}$. The new form of the terminal equations of resistors is

$$\begin{bmatrix} I_3 \\ I_4 \end{bmatrix} = D_r \left(\begin{bmatrix} V_3 \\ V_4 \end{bmatrix} \right) \qquad 2.9.4$$

For the purpose of this section, Eq. 2.9.4 contains only two-terminal resistor characteristics, and therefore the ith component of the function D_r depends only on the ith component of $\begin{bmatrix} V_3 \\ V_4 \end{bmatrix}$. However, it is possible to group multiterminal resistors also into Eq. 2.9.4, in which case the above condition will not hold.

From Eq. 2.9.4 we can obtain equations similar to Eq. 2.8.11. These equations are :

$$\begin{bmatrix} U & A_{31} \end{bmatrix} D_r\left(\begin{bmatrix} 0 \\ -B_{12} \end{bmatrix} V_2 + \begin{bmatrix} 0 \\ -B_{11} \end{bmatrix} V_1 + \begin{bmatrix} U \\ -B_{13} \end{bmatrix} V_3 \right)$$

$$+A_{32}\, I_5 + A_{33}\, I_6 = \underline{0} \qquad 2.9.5$$

Methods for solving Eq. 2.9.5 have been given by (3,4). However, the conditions imposed by these authors are restrictive. They are :

For the typical resistor with a characteristics of the form of Eq. 2.6.1(b), the following should be satisfied :

$$\text{(i)} \quad v_1 > v_2 \Rightarrow G(v_1) > G(v_2) \qquad 2.9.6$$

$$\text{(ii)} \quad M_1 \left| v_1 - v_2 \right| \leq \left| G(v_1) - G(v_2) \right| \leq M_2 \left| v_1 - v_2 \right| \qquad 2.9.7$$

for some positive, finite M_1 and M_2 and all v_1 and v_2.

These conditions can be stated in a short form by saying that each resistor should have a strictly monotonically increasing

characteristic with bounded slope. Since in such a case it is always possible to write the characteristics in the form of Eq. 2.6.1 (b), it is not necessary to consider the other form. The proof for the above result can be found in (3).

Characteristics involving negative resistance, such as a tunnel diode characteristic, can not be handled by the algorithms of (3,4).

In such cases, the characteristics can be written in only one of the two forms represented by Eqs. 2.6.1(a), 2.6.1(b). For example, the tunnel diode characteristics can only be written explicitly in the current variable. Components which do not satisfy the conditions of Eqs. 2.9.6 , 2.9.7 can be handled if :

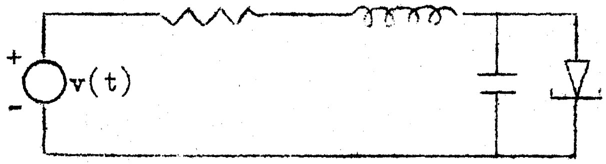
(i) The variable in which the characteristics is explicit can be made the primary variable; and

(ii) The secondary variable of the component can be determined from the knowledge of the primary variables of drivers, capacitors and inductors only.

If the above conditions are satisfied, the component can be represented by a dependent driver controlled by its own secondary variable. Figure 2.9.1 shows an example of a network in which a tunnel diode is represented as a dependent current driver.

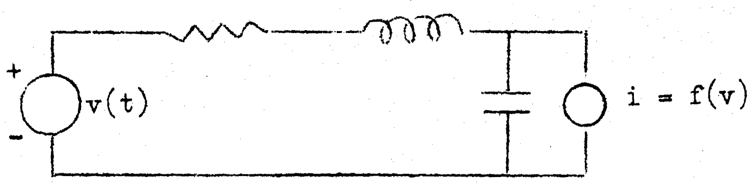
2.10 Small-Signal Characteristics of Nonlinear Components :

In chapter 4 we shall make use of the small-signal behaviour of nonlinear components. The small-signal behaviour of components is described in terms of the matrices D_d^s and D_r^s



(a)

A network containing a tunnel diode



(b)

The network of (a), with the tunnel diode represented as a dependent current driver.

Fig. 2.9.1 A network containing a tunnel diode.

defined as follows :

$$D_d^s = \left[\frac{\partial d_{d_i}}{\partial x_j} \right] \qquad 2.10.1$$

where

$$X = \begin{bmatrix} Q_2 \\ \\ \phi_1 \end{bmatrix} \qquad 2.10.2$$

As pointed out in the previous section, the ith component of D_d depends only on the ith component of X. Therefore, D_d is diagonal.

The matrix D_r^s is defined in terms of a more general form of the resistor characteristics than Eq. 2.9.4. This form is

$$X(t) = D_r \, (Y(t)) \qquad 2.10.3$$

where X and Y are complementary vectors of voltages and currents, as explained earlier. From this, the small-signal characteristics D_r^s is defined as :

$$D_r^s = \left[\frac{\partial d_{r_i}}{\partial y_j} \right] \qquad 2.10.4$$

If Eq. 2.10.3 contains some multiterminal components then the matrix D_r^s may not be diagonal. If, however, D_r contains only two-terminal components then the ith component of D_r depends only on the ith component of Y and therefore D_r^s is diagonal.

2.11 Additional Remarks

In sections 2.8 and 2.9 we have indicated the conditions under which the solution of a network can be ensured. These conditions

are more general than those appearing in earlier work; this improvement is a consequence of the discussion on the inclusion of dependent drivers given in those sections. Based on this discussion the development of a computer programme is currently under progress, which on successful completion is expected to have broader applicability than existing programmes for similar analysis. Specifically, negative resistance characteristics are incorporated in the computer analysis under conditions where solutions are assured.

3. Sensitivity Studies.

3.1 Introduction.

In this chapter we consider procedures for determining the additional information about the network regarding the changes in the response variables with respect to

(a) component parameters, and

(b) initial conditions.

The models for sensitivity studies that are developed here admit very useful network interpretations. Consequently, the analysis programme that has been referred to in the previous chapter can also be used for sensitivity studies.

Although time-domain notations are used throughout in the following study, it may be noted that this fact has nowhere been used for the development of the models. Consequently, the models developed are equally applicable to time- and frequency-domain analyses. The nature of the models is convenient for use in both analytical as well as computer analyses.

First-order sensitivity models were indicated by Leeds(9). Similar, but more general models were derived for state equations (20)

The formulation procedure given here is convenient for computer analysis of first-order sensitivity. The procedure has been extended to higher-order sensitivities. A simple recursive algorithm is possible for the linear case.

We shall consider time-invariant networks. Linear and nonlinear networks are discussed separately.

3.2 Graph of the Sensitivity Model.

In the previous chapter, we have discussed a procedure for determining the response of a network, which consists of the voltages $V(t)$ and currents $I(t)$. Now we turn our attention to the variables in the vectors

$$\frac{\partial V}{\partial p}(t) \text{ and } \frac{\partial I}{\partial p}(t), \text{ and also}$$

$$\frac{\partial^k V(t)}{\partial p_1 \, \partial p_2 \cdots \partial p_k} \quad \text{and} \quad \frac{\partial^k I(t)}{\partial p_1 \, \partial p_2 \cdots\cdots \partial p_k} .$$

In the above expressions, the symbols p, p_1, $p_2, \ldots, p_k$ denote the parameters of the network. They may also represent initial conditions of the network. In the linear case, they could be values of the components, i.e. R, L, C, etc. The component parameters for nonlinear components will be defined later.

For both linear and nonlinear networks, we have the f-cutset and f-circuit equations,

$$A\, I(t) = \underline{0} \;,$$

and

$$B\, V(t) = \underline{0} \qquad 3.2.1$$

Differentiating Eq. 3.2.1 partially with respect to a parameter p of the network, we can write :

$$A \frac{\partial I(t)}{\partial p} = \underline{0}$$

and

$$B \frac{\partial V(t)}{\partial p} = \underline{0} \qquad 3.2.2$$

The similarity of form between Eqs. 3.2.1 and 3.2.2 indicates that if components with appropriate characteristics are used, then it is possible to obtain the sensitivity coefficients

$$\frac{\partial I}{\partial p}(t) \text{ and } \frac{\partial V}{\partial p}(t)$$

as the solution to a network having the same graph as the given network.

3.3 Sensitivity Model for Linear Networks :

We shall consider the sensitivity model for each type of component separately. In the end of the section, we shall sum up the results in the form of a table.

Independent drivers.

The terminal equations of independent drivers are :

$$v = v(t) \tag{3.3.1}$$

for a voltage driver, and

$$i = i(t) \tag{3.3.2}$$

for a current driver.

From these we get

$$\frac{\partial v}{\partial p}(t) = 0 \tag{3.3.3}$$

and

$$\frac{\partial i}{\partial p}(t) = 0 \, . \tag{3.3.4}$$

The counterparts of independent drivers in the sensitivity model have terminal equations of the form of Eqs. 3.3.3 and 3.3.4.

In other words, the counterpart of a voltage driver is a voltage driver of zero value (a short circuit), and the counterpart of a current driver is a current driver of zero value (an open circuit).

Resistors.

The typical resistor characteristic is modelled by the equation

$$v(t) = R\, i(t) \tag{3.3.5}$$

Differentiating Eq. 3.3.5 partially with respect to a parameter p which is different from R and G = 1/R, we get

$$\frac{\partial v}{\partial p}(t) = R \frac{\partial i}{\partial p}(t) \tag{3.3.6}$$

Thus, if the parameter p does not refer to the resistor of Eq. 3.3.5, then the counterpart of a resistor in the sensitivity model is a similar resistor.

Next, let us find the counterpart of the resistor if the sensitivity terms

$$\frac{\partial V(t)}{\partial R} \quad \text{and} \quad \frac{\partial I(t)}{\partial R}$$

are to be found with respect to the parameter R. From Eq. 3.3.5 we obtain

$$\frac{\partial v}{\partial R}(t) = R\frac{\partial i}{\partial R}(t) + i(t) \tag{3.3.7}$$

A nonlinear resistor can be described by a terminal equation such as Eq. 3.3.7. Thus, using nonlinear network analysis methods, the resulting network can be solved.

Alternatively, we can rewrite Eq. 3.3.7 as

$$\frac{\partial v}{\partial R}(t) = R\left(\frac{\partial i}{\partial R}(t) + \frac{i(t)}{R}\right) \qquad 3.3.8$$

Equation 3.3.8 can be modelled by a parallel combination of a resistor of value R and a current driver of value $i(t)/R$. The direction of the driver is reverse of the direction in which the current $i(t)$ is measured in the original network. The combination modelling Eq. 3.3.8 is shown in Figure 3.3.1(a).

For finding the counterpart of the resistor in the case where the sensitivity coefficients

$$\frac{\partial V(t)}{\partial G} \quad \text{and} \quad \frac{\partial I(t)}{\partial G}$$

are to be determined, we first re-write the terminal equation of the resistor as :

$$i(t) = G\, v(t) \qquad 3.3.9$$

Differentiating Eq. 3.3.9 with respect to the parameter G we get

$$\frac{\partial i(t)}{\partial G} = G\,\frac{\partial v}{\partial G}(t) + v(t) \qquad 3.3.10$$

As in the case of the model discussed before, Eq. 3.3.8 can be treated as the terminal equation of a nonlinear resistor.

Alternatively, it can be modelled by the series combination of a voltage driver of value $v(t)/G$ and a resistor of conductance G. The direction of the voltage driver in this model is the reverse of the direction in which the voltage $v(t)$ is measured. The model is shown in Figure 3.3.1(b).

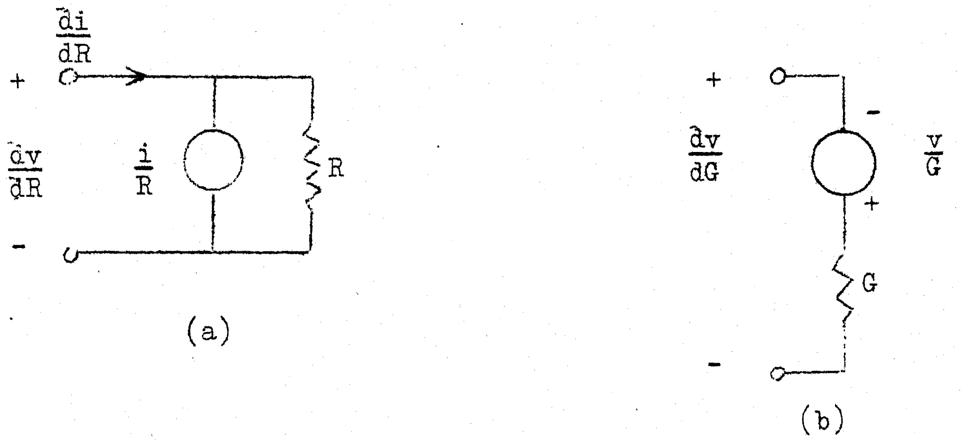


Fig. 3.3.1 Sensitivity models of a resistor.

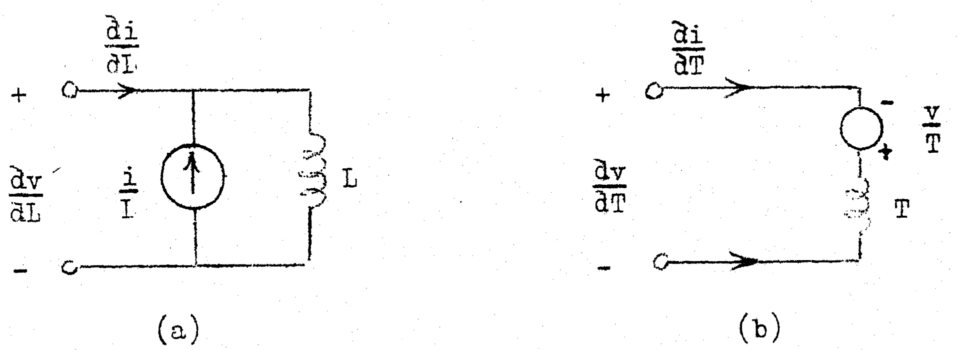


Fig. 3.3.2 Sensitivity models of an inductor.

Inductor.

For an inductor we have

$$v(t) = L\frac{di}{dt} \tag{3.3.11}$$

When the sensitivity pertains to a parameter which refers to a component other than the inductor of Eq. 3.3.11, we have

$$\frac{\partial v(t)}{\partial p} = L\frac{d}{dt}\left(\frac{\partial i}{\partial p}\right) \tag{3.3.12}$$

In this case the counterpart of the inductor in the sensitivity model is a similar inductor.

In addition to the terminal equation of the inductor in the sensitivity model, we must also determine its initial condition for a complete specification of the sensitivity model.

Since p does not refer to the inductor of Eq. 3.3.11, we have

$$\frac{\partial i(0)}{\partial p} = 0 \tag{3.3.13}$$

Hence, the initial condition on the counterpart of the inductor in the sensitivity model is zero.

If the coefficients

$$\frac{\partial V}{\partial L}(t) \text{ and } \frac{\partial I}{\partial L}(t)$$

are to be determined, we have from Eq. 3.3.11,

$$\frac{\partial v}{\partial L} = L\frac{d}{dt}\left(\frac{\partial i}{\partial L}\right) + \frac{di}{dt} \tag{3.3.14}$$

Equation 3.3.14 can be rewritten in the form

$$\frac{\partial v}{\partial L} = \frac{d}{dt}\left(\frac{\partial \phi}{\partial L}\right)$$

and

$$\frac{\partial i}{\partial L} = \frac{\frac{\partial \phi}{\partial L} - i}{L} \qquad 3.3.15$$

Equation 3.3.15 can be modelled by a nonlinear inductor.

Alternatively, Eq. 3.3.14 can be rewritten as

$$\frac{\partial v}{\partial L} = L\frac{d}{dt}\left(\frac{\partial i}{\partial L} + \frac{i}{L}\right) \qquad 3.3.16$$

Equation 3.3.16 can be modelled by a parallel combination of an inductor and a current driver, shown in Figure 3.3.2(a).

For the coefficients

$$\frac{\partial V}{\partial T}(t) \text{ and } \frac{\partial I}{\partial T}(t)$$

(where $T = 1/L$), a similar analysis can be carried out to arrive at the model shown in Figure 3.3.2(b).

Initial Conditions :

We must determine the initial condition on the counterpart of an inductor in a sensitivity model in order that the latter model is completely deterministic. When the value of the inductor is varied infinitesimally, one may decide either to hold the initial current invariant, or to hold the initial flux invariant. The initial condition on the inductor in the sensitivity model depends on this decision.

If the current is invariant, then clearly we have

$$\frac{\partial i(0)}{\partial L} = 0 \qquad 3.3.17$$

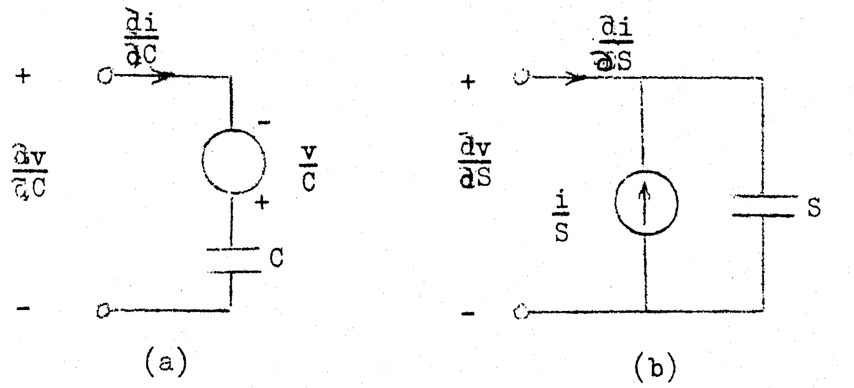


Fig. 3.3.3 Sensitivity models of a capacitor.

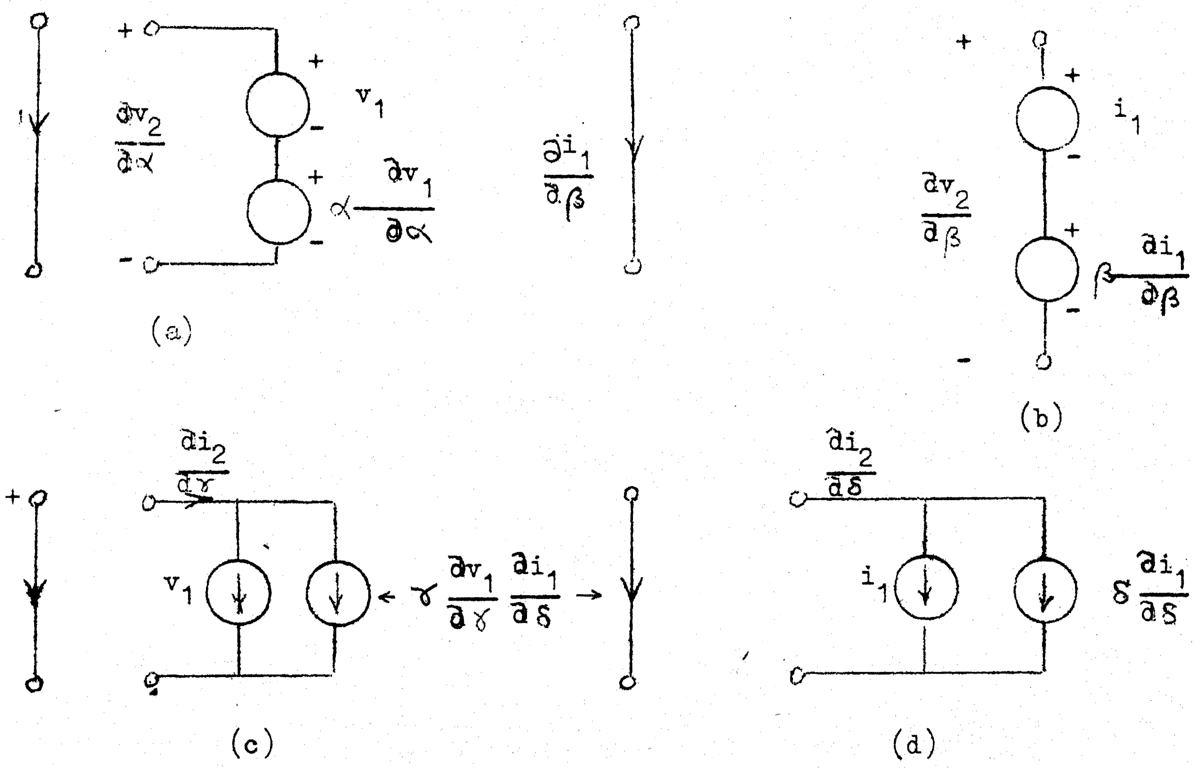


Fig. 3.3.4 Sensitivity models of dependent drivers.

for the model in Figure 3.3.2(a), or

$$\frac{\partial\, i(0)}{\partial\, T} = 0 \qquad 3.3.18$$

for the model in figure 3.3.2(b).

If on the other hand the flux is invariant, then we have

$$\phi(0) = L\, i(0) = \text{constant}.$$

Hence,

$$\frac{\partial\, i(0)}{\partial\, L} = -\frac{i(0)}{L} \qquad 3.3.19$$

for the model of Figure 3.3.2(a).

For the model of Figure 3.3.2(b), we have

$$i(0) = T\,\phi(0).$$

Hence

$$\frac{\partial\, i(0)}{\partial\, T} = \phi(0) = \frac{i(0)}{T}\ . \qquad 3.3.20$$

Capacitors.

The typical terminal equation of a capacitor is

$$i(t) = C\,\frac{dv}{dt} \qquad 3.3.21$$

It can be shown as in the case of components discussed earlier, that the counterpart of the capacitor of Eq. 3.3.21 in the sensitivity model, when the parameter does not pertain to the capacitor, is a similar capacitor.

When the coefficients

$$\frac{\partial\, V(t)}{\partial\, C} \quad \text{and} \quad \frac{\partial\, I(t)}{\partial\, C}$$

are to be determined, we have

$$\frac{\partial\, i}{\partial\, C}(t) = C\, \frac{d}{dt}\left(\frac{\partial\, v}{\partial\, C}\right) + \frac{dv}{dt}$$

$$= C\, \frac{d}{dt}\left(\frac{\partial\, v}{\partial\, C} + v\right) \qquad 3.3.22$$

Equation 3.3.22 can be modelled by a nonlinear **capacitor** when rewritten suitably. Alternatively, it can be modelled by the series combination of a capacitor and a voltage driver, shown in Figure 3.3.3(a).

When the coefficients

$$\frac{\partial\, V}{\partial\, S}(t) \quad \text{and} \quad \frac{\partial\, I}{\partial\, S}(t)$$

(where $S = 1/C$) are to be determined, the model shown in Figure 3.3.3(b) can be derived in a similar manner.

As in the case of an inductor, the initial condition is required for a complete description of the sensitivity model.

When the parameter p does not refer to the capacitor of Eq. 3.3.21, we have

$$\frac{\partial\, v(0)}{\partial\, p} = 0. \qquad 3.3.23$$

When the parameter refers to the capacitor , two possibilities arise. We may desire to keep either the initial voltage or the initial charge constant. If the initial voltage is invariant, we have

$$\frac{\partial\, v(0)}{\partial\, C} = 0 \qquad 3.3.24$$

for the model of Figure 3.3.3(a), and

$$\frac{\partial v(0)}{\partial S} = 0 \tag{3.3.25}$$

for the model of Figure 3.3.3(b).

When the initial charge is invariant, we have

$$v(0) = \frac{q(0)}{C} = S\, q(0);$$

and hence

$$\frac{\partial v(0)}{\partial C} = -\frac{v(0)}{C} \tag{3.3.26}$$

for the model of Figure 3.3.3(a), and

$$\frac{\partial v(0)}{\partial S} = \frac{v(0)}{S}$$

for the model of Figure 3.3.3(b).

Consider the case where the initial current through an inductor or the initial voltage across a capacitor is the parameter. In this case, we can derive the terminal equations of the counterparts of inductors and capacitors from Eqs. 3.3.11 and 3.3.21 respectively. These counterparts are linear inductors and capacitors identical to the ones in Eqs. 3.3.11 and 3.3.21. The initial conditions on these counterparts are a unit current and a unit voltage, respectively, since

$$\frac{\partial i(0)}{\partial i(0)} = 1 \tag{3.3.27}$$

for an inductor, and

$$\frac{\partial v(0)}{\partial v(0)} = 1 \qquad 3.3.28$$

for a capacitor.

Dependent drivers.

We restate the equations of the four types of dependent drivers here for convenience. They are :

Type 1

$$v_2(t) = \alpha\, v_1(t) \qquad 3.3.29$$

Type 2

$$v_2(t) = \beta\, i_1(t) \qquad 3.3.30$$

Type 3

$$i_2(t) = \gamma\, v_1(t)$$

Type 4

$$i_2(t) = \delta\, i_1(t) \qquad 3.3.32$$

When sensitivity coefficients with respect to a parameter not referring to a dependent driver are to be determined, it can be shown that the counterpart of a dependent driver is a similar dependent driver. We shall discuss the models of the counterparts of dependent drivers in sensitivity models for their own parameters.

Type 1

From Eq. 3.3.29 we have

$$\frac{\partial v_2(t)}{\partial \alpha} = \alpha \frac{\partial v_1}{\partial \alpha}(t) + v_1(t) \qquad 3.3.33$$

Equation 3.3.33 can be represented as the series combination of a dependent driver and an independent driver of value v_1. Such a model is shown in Figure 3.3.4(a).

Sensitivity models for dependent driver types 2,3 and 4 can be derived in a similar manner. These models are shown in Figures 3.3.4(b, c,d) respectively.

The results of this section can be summarized by the following algorithm :

Algorithm 3.3.1

For obtaining a network whose response consists of the sensitivity coefficients

$$\frac{\partial V}{\partial p}(t) \text{ and } \frac{\partial I}{\partial p}(t)$$

where p could assume any one of the parameters R,L.C,T,S, G, α , β , γ , or δ , or one of the initial conditions i(0) or v(0) , go through the following steps :

(i) Short-circuit all independent voltage drivers and open-circuit all independent current drivers of the original network

(ii) Connect the appropriate driver in parallel or series with the component whose parameter is p, and / or set the appropriate initial condition on that component as indicated by Table 3.3.1. Set all other initial conditions to zero.

3.4 Higher-order Sensitivities.

General higher-order sensitivity coefficients of the type

$$\frac{\partial^k V(t)}{\partial p_1 \partial p_2 \cdots \partial p_k} \text{ and } \frac{\partial^k I(t)}{\partial p_1 \partial p_2 \cdots \partial p_k}$$

will be discussed in this section. On the basis of the results obtained in the last section, we can straightaway conclude that independent voltage and current drivers in the original network are

Table 3.3.1 First-order Sensitivity models for linear components.

Component		Driver			Initial Condition	
Type	parameter	Type	Placing	Value	i or v invariant	q or ϕ invariant
Resistor	R	Current	Parallel	i/R	-	-
"	G	Voltage	Series	v/G	-	-
Inductor	L	Current	Parallel	i/L	0	-io/L
"	T	Voltage	Series	v/T	0	io/T
" or capacitor	io or v_o	None	-	-	1	
Capacitor	C	Voltage	Series	v/C	0	-vo/C
"	S	Current	Parallel	i/S	0	vo/S
Dependent Driver Type 1	α	Voltage	Series	v_1	-	-
Type 2	β	Voltage	Series	i_1	-	-
Type 3	γ	Current	Parallel	v_1	-	-
Type 4	δ	Current	Parallel	i_1	-	-

replaced by short and open circuits, respectively, in the sensitivity network. Similarly it is quite obvious that if the parameters p_1 - - - - p_k do not refer to a certain component of the R, L, C, or dependent driver types, then the counterpart of such a component is a similar component; and in the case of a capacitor or an inductor the initial condition is zero.

We shall now derive models for the cases when one of the parameters refers to the component. For example, in the case of a resistor whose terminal equation is

$$v(t) = R\, i(t) \qquad 3.4.1$$

Suppose it is desired to find the sensitivities

$$\frac{\partial^k V}{\partial R\ \partial p_2 ---- \partial p_k} \quad \text{and} \quad \frac{\partial^k I}{\partial R\, \partial p_2 ---- \partial p_k}$$

where none of p_2, - - - p_k refers to the resistor of Eq. 3.4.1. We have from Eq. 3.4.1 :

$$\frac{\partial^k v}{\partial R\, \partial p_2 ---- \partial p_k} = R \frac{\partial^k i}{\partial R\, \partial p_2 ----- \partial p_k} + \frac{\partial^{k-1} i}{\partial p_2 -------- \partial p_k} \qquad 3.4.2$$

As a more general case, consider the following sensitivity coefficients

$$\frac{\partial^k V}{\partial R^{k_1} \partial p_{k_1+1} ----- \partial p_k} \quad \text{and} \quad \frac{\partial^k I}{\partial R^{k_1} \partial p_{k1+1} ----- \partial p_k}$$

in which the parameter R appears k_1 times.

Again, from Eq. 3.4.1 we have

$$\frac{\partial^k v}{\partial R^{k_1} \partial p_2 --- \partial p_k} = R \frac{\partial^k i}{\partial R^{k_1} \partial p_2 --- \partial p_k} + k_1 \frac{\partial^{k-1} i}{\partial R^{k_1-1} \partial p_2 -- \partial p_k} \qquad 3.4.3$$

Equations 3.4.2 are a special case of Eq. 3.4.3 where $k_1 = 1$.

Equations 3.4.2 and 3.4.3 are similar to Eq. 3.3.7 of the previous section. The model shown in Figure 3.3.1(a) can be used to model Eqs. 3.4.2 and 3.4.3, when appropriate values of the driver are used. Similar equations and models can easily be derived for inductors, capacitors, and dependent drivers. Models shown in Figures 3.3.1 to 3.3.4 can be used with the values of drivers given at the end of this section in Table 3.4.1.

Initial conditions :

Here once again we have the choice of holding either the flux or the current invariant (in the case of an inductor).

If the current is held invariant then the initial condition on that inductor is zero. Similarly, if the voltage of a capacitor is held invariant, the initial condition on that capacitor is zero.

If the flux of an inductor is held invariant, and if sensitivity coefficients of the kind

$$\frac{\partial^k V(t)}{\partial L^k} \quad \text{and} \quad \frac{\partial^k I(t)}{\partial L^k}$$

are sought, then we have

$$\frac{\partial^k i(0)}{\partial L^k} = \frac{\partial^k}{\partial L^k}\left(\frac{\phi_o}{L}\right) = \frac{(-1)^k\, k\, .\, i(0)}{L^k} \qquad 3.4.4$$

If the sensitivity coefficients are sought with respect to any parameter other than L, the initial condition on the inductor is zero. Similarly, if sensitivity coefficients of the type

$$\frac{\partial^k V}{\partial C^k}(t) \text{ and } \frac{\partial^k I}{\partial C^k}\,(t)$$

are sought, and the charge of the capacitor is held invariant, we have

$$\frac{\partial^k v(0)}{\partial C^k} = \frac{\partial^k}{\partial C^k}\left(\frac{q}{C}(0)\right)$$

$$= \frac{(-1)^k\, k\, .\, v(0)}{C^k} \qquad 3.4.5$$

If sensitivity coefficients with respect to T in the case of an inductor or S in the case of a capacitor are sought, we have a simpler situation. We have in particular,

$$\frac{\partial^k i(0)}{\partial T^k} = 0 \quad \text{for } k > 1; \qquad 3.4.6$$

and similarly

$$\frac{\partial^k v(0)}{\partial S^k} = 0 \quad \text{for } k > 1. \qquad 3.4.7$$

In connection with sensitivity coefficients of the type

$$\frac{\partial^k V(t)}{\partial p_1 - - - \partial p_k} \quad \text{and} \quad \frac{\partial^k I(t)}{\partial p_1 - - - \partial p_k}$$

where $k > 1$, if one of the parameters $p_1, \ldots, p_k$ is an initial condition, the corresponding inductor or capacitor has no driver associated with it and a zero initial condition. In other words, the sensitivity model of that component is just like the case where the parameters do not refer to that component. If two of the parameters are initial conditions, then the coefficients sought are all zero.

In the above analysis, we have found a method of finding the kth order sensitivity coefficients based on the knowledge of some coefficients of order (k-1). However, this does not pose a serious problem since the first-order coefficients can be found with the knowledge of only the network response, and then the second-order coefficients can be found using the first-order ones, and so on, upto any desired order.

Let us summarize this section again in the form of a table and an algorithm for finding

$$\frac{\partial^k V}{\partial p_1 \partial p_2 \cdots \partial p_k} \quad \text{and} \quad \frac{\partial^k I}{\partial p_1 \partial p_2 \cdots \partial p_k}, \text{ where } k \quad 1.$$

Algorithm 3.4.1 (i) Short-circuit all independent voltage drivers, and open-circuit all independent current drivers.

(ii) Set the initial conditions on all dynamic components except the ones whose parameters are in $p_1, p_2, \ldots, p_k$, to zero.

(iii) Connect the appropriate drivers in series or parallel with the components whose parameters are in $p_1, \ldots, p_k$, and/or set the appropriate initial condition on that component, as indicated by Table 3.4.1.

Table 3.4.1 Higher-order Sensitivity models for linear components.

Component		Associated Driver			Initial Condition	
Type	Parameter	Type	Placing	Value	i or v invariant	ϕ or q invariant or i or v parameter
Resistor	R	Current	Parallel	$\frac{k_1}{R}\frac{\partial^{k-1} i}{\partial p_2 \cdots \partial p_k}$	-	-
"	G	Voltage	Series	$\frac{k_1}{G}\frac{\partial^{k-1} v}{\partial p_2 \cdots \partial p_k}$	-	-
Inductor	L	Current	Parallel	$\frac{k_1}{L}\frac{\partial^{k-1} i}{\partial p_2 \cdots \partial p_k}$	0	$(-1)^k(k-1).\frac{i_o}{L^k}$
"	T	Voltage	Series	$\frac{k_1}{T}\frac{\partial^{k-1} v}{\partial p_2 \cdots \partial p_k}$	0	$\frac{i_o}{T}$ if $k=1$, 0 if $k>1$
" or Capacitor	i_o or v_o	None	-	-	-	0
Capacitor	C	Voltage	Series	$\frac{k_1}{C}\frac{\partial^{k-1} v}{\partial p_2 \cdots \partial p_k}$	0	$\frac{(-1)^k(k-1).v_o}{C^k}$
"	S	Current	Parallel	$\frac{k_1}{S}\frac{\partial^{k-1} v}{\partial p_2 \cdots \partial p_k}$	0	$\frac{v_o}{S}$ if $k=1$, 0 if $k>1$
Dependent Driver Type 1		Voltage	Series	$k_1 \frac{\partial^{k-1} v_1}{\partial p_2 \cdots \partial p_k}$	-	-
Type 2		Voltage	Series	$k_1 \frac{\partial^{k-1} i_1}{\partial p_2 \cdots \partial p_k}$	-	-
Type 3		Current	Parallel	$k_1 \frac{\partial^{k-1} v_1}{\partial p_2 \cdots \partial p_k}$	-	-
Type 4		Current	Parallel	$k_1 \frac{\partial^{k-1} i_1}{\partial p_2 \cdots \partial p_k}$	-	-

Example 3.4.1

For the triode amplifier shown in Figure 3.4.1 a find

$$A = \frac{v_2}{v_1}, \quad \frac{\partial A}{\partial \mu}, \quad \frac{\partial A}{\partial R_L}, \quad \frac{\partial^2 A}{\partial R_L{}^2}.$$

First, to simplify, we connect a unit voltage driver at the input.

Then,

$$v_2 = A.$$

The equivalent network is shown in Figure 3.4.1b. The sensitivity models for $\frac{\partial A}{\partial \mu}$, $\frac{\partial A}{\partial R_L}$ and $\frac{\partial^2 A}{\partial R_L{}^2}$ are shown in Figure 3.4.1 c,d, and e respectively. Solving these we obtain

$$A = -\mu \frac{R_L}{R_p + R_L};$$

$$\frac{\partial A}{\partial \mu} = -\frac{R_L}{R_p + R_L};$$

$$\frac{\partial A}{\partial R_L} = -\mu \frac{R_p}{(R_p + R_L)^2};$$

and $$\frac{\partial^2 A}{\partial R_L{}^2} = \frac{2 \mu R_p}{(R_p + R_L)^3}.$$

3.5 Nonlinear Network.

In the case of linear networks, the component values were used as parameters of the network. No such unique and natural association is possible in the nonlinear case. For nonlinear networks, a parameter (always denoted by p) is introduced in the terminal

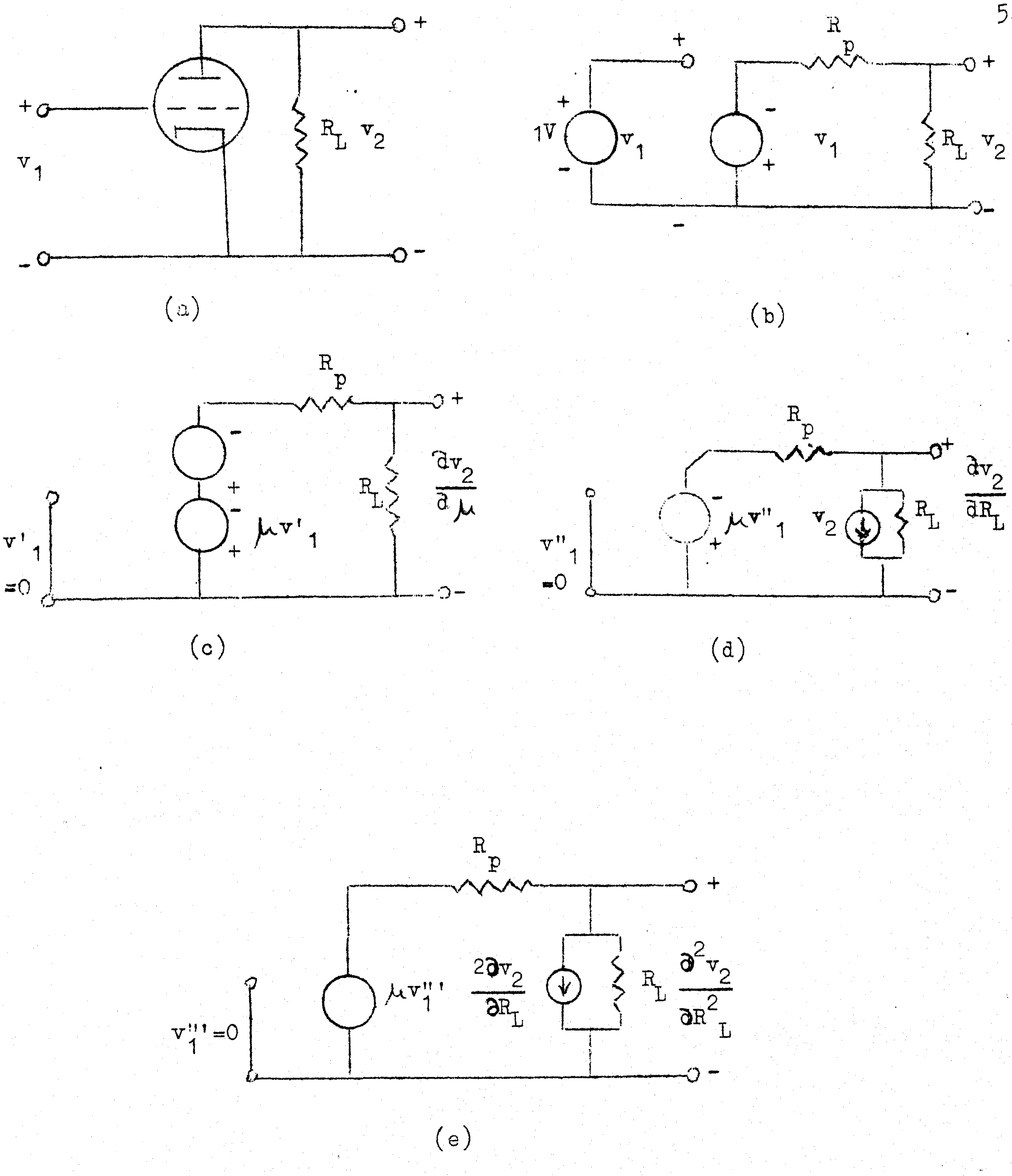


Fig. 3.4.1 : Sensitivity of a triode amplifier.

equation of each type of component other than independent drivers. The relation of this parameter to the function in the terminal equation is not prescribed. The sensitivity coefficients are then described in terms of the parameter p. Consequently, the models involve the partial derivatives of the functions in the terminal equations of nonlinear components with respect to p . The component characteristics are modelled by the following equations when the parameter p is introduced :

Resistor : $$i = G(v,p) \tag{3.5.1}$$

Inductor :

$$v = \frac{d\phi}{dt}, \qquad i = T(\phi,p) \tag{3.5.2}$$

Capacitor :

$$i = \frac{dq}{dt}, \qquad v = S(q, p) \tag{3.5.3}$$

Dependent Drivers

Type 1 : $$v_2 = V(v_1,p) \tag{3.5.4}$$

Type 2 : $$v_2 = V(i_1, p) \tag{3.5.5}$$

Type 3 . $$i_2 = I(v_1, p) \tag{3.5.6}$$

Type 4 : $$i_2 = I(i_1, p) \tag{3.5.7}$$

From these we obtain the sensitivity models as follows :

Resistor :

$$\frac{\partial i}{\partial p'} = \frac{\partial G}{\partial v} \cdot \frac{\partial v}{\partial p'} \qquad 3.5.8$$

where p' does not refer to that resistor.

For its own parameter, we have

$$\frac{\partial i}{\partial p} = \frac{\partial G}{\partial v} \cdot \frac{\partial v}{\partial p} + \frac{\partial G}{\partial p} \qquad 3.5.9$$

In the above models we have seen that the knowledge of the derivatives

$$\frac{\partial G}{\partial v} \text{ and } \frac{\partial G}{\partial p}$$

pertaining to the characteristic of the resistor are required for its sensitivity models. If second-order sensitivity coefficients were sought, we have the following equation :

$$\frac{\partial^2 i}{\partial p \partial p'} = \frac{\partial G}{\partial v} \cdot \frac{\partial^2 v}{\partial p \partial p'} + \frac{\partial^2 G}{\partial v^2} \cdot \frac{\partial v}{\partial p} \cdot \frac{\partial v}{\partial p'} + \frac{\partial^2 G}{\partial p \partial v} \cdot \frac{\partial v}{\partial p} ,$$

3.5.10

Thus, for each increase in the order of the sensitivity coefficients, additional information about the component terminal equations is required for the sensitivity model.

We shall abandon the study of higher-order sensitivities of nonlinear networks, since it is impractical to require so much information about any network which is big enough to justify computer solution. However, there is no theoretical difficulty in constructing models for higher-order sensitivity coefficients.

Let us continue the discussion on first-order models.

For an inductor, when p' is a parameter not refering to it, we have

$$\frac{\partial v}{\partial p'}(t) = \frac{d}{dt}\left(\frac{\partial \phi}{\partial p'}\right)$$

$$\frac{\partial i(t)}{\partial p'} = \frac{\partial T}{\partial \phi} \cdot \frac{\partial \phi(t)}{\partial p'} \qquad 3.5.11$$

And for its own parameter p,

$$\frac{\partial v}{\partial p} = \frac{d}{dt}\left(\frac{\partial \phi}{\partial p}\right)$$

and 3.5.12

$$\frac{\partial i}{\partial p} = \frac{\partial T}{\partial p} + \frac{\partial T}{\partial \phi} \cdot \frac{\partial \phi}{\partial p}$$

For a capacitor, for p' which is not its own parameter,

$$\frac{\partial i}{\partial p'} = \frac{d}{dt}\left(\frac{\partial q}{\partial p'}\right)$$

and 3.5.13

$$\frac{\partial v}{\partial p'} = \frac{\partial S}{\partial q} \cdot \frac{\partial q}{\partial p'}$$

And for its own parameter :

$$\frac{\partial i}{\partial p} = \frac{d}{dt}\left(\frac{\partial q}{\partial p}\right)$$

and 3.5.14

$$\frac{\partial v}{\partial p} = \frac{\partial S}{\partial p} + \frac{\partial S}{\partial q} \cdot \frac{\partial q}{\partial p}$$

For the four types of dependent drivers we obtain the following equations, where p is the parameter of the driver concerned and p' a different parameter. We have :

Type 1:

$$\frac{\partial v_2}{\partial p'} = \frac{\partial V}{\partial v_1} \cdot \frac{\partial v_1}{\partial p'} \qquad 3.5.15$$

or $$\frac{\partial v_2}{\partial p} = \frac{\partial V}{\partial p} + \frac{\partial V}{\partial v_1} \cdot \frac{\partial v_1}{\partial p} \qquad 3.5.16$$

Type 2 :

$$\frac{\partial v_2}{\partial p'} = \frac{\partial V}{\partial i_1} \cdot \frac{\partial i_1}{\partial p'} \qquad 3.5.17$$

or $$\frac{\partial v_2}{\partial p} = \frac{\partial V}{\partial p} + \frac{\partial V}{\partial i_1} \cdot \frac{\partial i_1}{\partial p} \qquad 3.5.18$$

Type 3 :

$$\frac{\partial i_2}{\partial p'} = \frac{\partial I}{\partial v_1} \cdot \frac{\partial v_1}{\partial p'} \qquad 3.5.19$$

or $$\frac{\partial v_2}{\partial p} = \frac{\partial I}{\partial p} + \frac{\partial I}{\partial v_1} \cdot \frac{\partial v_1}{\partial p} \qquad 3.5.20$$

Type 4 :

$$\frac{\partial i_2}{\partial p'} = \frac{\partial I}{\partial i_2} \cdot \frac{\partial i_2}{\partial p'} \qquad 3.5.21$$

or $$\frac{\partial i_2}{\partial p} = \frac{\partial I}{\partial p} + \frac{\partial I}{\partial i_2} \cdot \frac{\partial i_2}{\partial p} \qquad 3.5.22$$

Initial Conditions :

Inductor If the flux is held invariant, then we have

$$\frac{\partial \phi_o}{\partial p} = 0 \qquad 3.5.23$$

If we try to hold the current invariant we get

$$\frac{\partial i}{\partial p} = \frac{\partial T}{\partial p} + \frac{\partial T}{\partial \phi_o} \frac{\partial \phi_o}{\partial p} = 0; \tag{3.5.24}$$

or
$$\frac{\partial \phi_o}{\partial p} = - \frac{\partial T/\partial p}{\partial T/\partial \phi_o} \tag{3.5.25}$$

provided $\frac{\partial T}{\partial \phi_o} \neq 0$;.which may not be true in general.

If ϕ_o is treated as a parameter and the sensitivity coefficients with respect to it sought, then we have

$$\frac{\partial \phi_o}{\partial \phi_o}(0) = 1 \; ; \tag{3.5.26}$$

and the terminal equation 3.5.10 is valid for that inductor.

Capacitor :

As in case of the inductor, it is not possible in general to hold the voltage invariant. If the charge is held invariant, we have

$$\frac{\partial q_o}{\partial p} = 0 \tag{3.5.27}$$

and provided $dS/dq_o \neq 0$ we can hold the voltage invariant by making

$$\frac{\partial v}{\partial p} = \frac{\partial S}{\partial p} + \frac{\partial S}{\partial q_o} \cdot \frac{\partial q_o}{\partial p} = 0 \tag{3.5.28}$$

or
$$\frac{\partial q_o}{\partial p} = - \frac{\partial S/\partial p}{\partial S/\partial q_o} \tag{3.5.29}$$

If q_o is treated as the parameter with respect to which the sensitivity is sought, then

$$\frac{\partial q}{\partial q_o}(0) = 1 \tag{3.5.30}$$

and the terminal equation 3.5.12 holds for that capacitor.

In conclusion, first-order sensitivity coefficients for fixed, nonlinear networks can be obtained as the solution to a time-varying network, all but one of whose components are linear, and the remaining one has a straight-line terminal characteristics not passing through the origin.

The following algorithm is a collection of the results of this section for purposes of application. It results in a network whose solution is $\frac{\partial V}{\partial p}$ and $\frac{\partial I}{\partial p}$, where p is the parameter of Eqs. 3.5.1 - 3.5.7 or an initial flux or an initial charge.

Algorithm 3.5.1 (i) Short-circuit all independent voltage drivers and open-circuit all independent current-drivers.

(ii) Replace all components except the one whose parameter is p by the one indicated in Table 3.5.1 and set initial conditions to zero.

(iii) Replace the component whose parameter is p and adjust its initial condition by the one indicated in Table 3.5.2.

Example 3.5.1

It is desired to study the rate of change of the amplitude of oscillation in the network of Figure 3.5.1 a with respect to a change in R. The characteristics of the tunnel diode G is given as :

$$i = - .02v + 0.00002\, v^3, \qquad 3.5.31$$

where i is in milliamperes and v in millivolts.

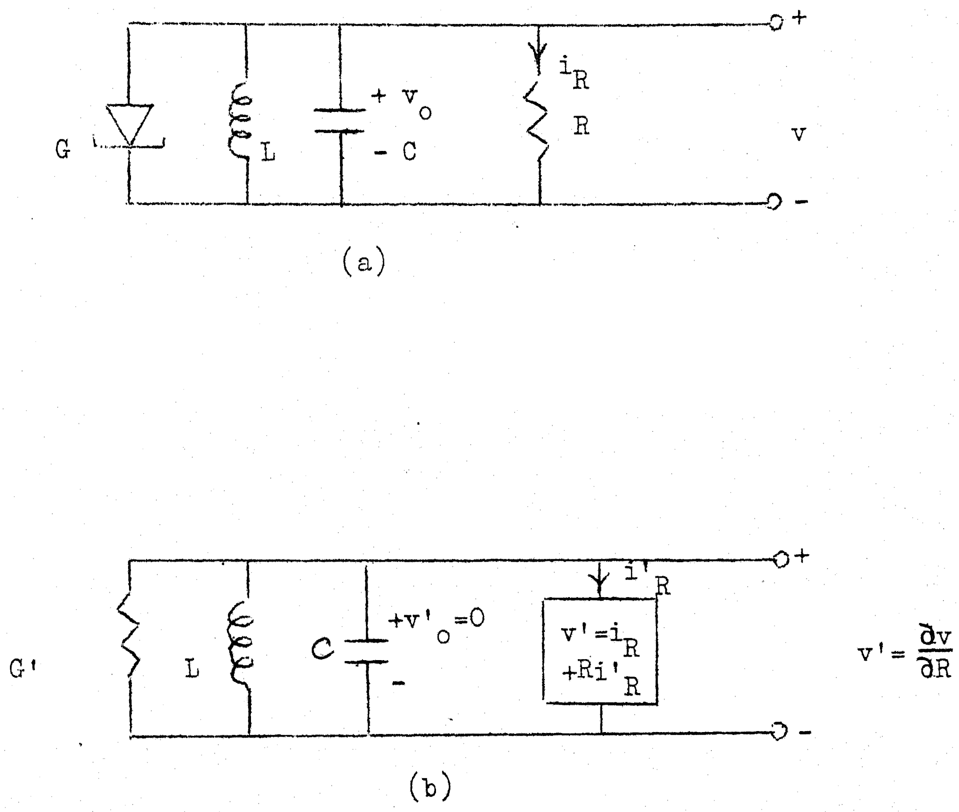


Fig. 3.5.1 Sensitivity of a negative-resistance oscillator.

Table 3.5.1

Component		Linear Replacement	
Type	Terminal Equation No.	Type	Terminal Equation No.
Resistor	3.5.1	Resistor	3.5.8
Inductor	3.5.2	Inductor	3.5.10
Capacitor	3.5.3	Capacitor	3.5.12
Dependent Driver Type 1	3.5.4	Dependent Driver Type 1	3.5.14
Dependent Driver Type 2	3.5.5	Dependent Driver Type 2	3.5.16
Dependent Driver Type 3	3.5.6	Dependent Driver Type 3	3.5.18
Dependent Driver Type 4	3.5.7	Dependent Driver Type 4	3.5.20

Table 3.5.2

Component			Replacement		Initial Condition		
Type	T.E. No.	Parameter	Type	T.E. No.	q_o or ϕ_o constant	i_o or v_o constant.	q_o or ϕ_o parameter
Resistor	3.5.1	p	Resistor	3.5.9	-	-	-
Inductor	3.5.2	p	Inductor	3.5.11	0	$-\dfrac{\partial T/\partial p}{\partial T/\partial \phi_o}$	-
"	"	ϕ_o	"	3.5.10	-	-	1
Capacitor	3.5.3	p	Capacitor	3.5.13	0	$-\dfrac{\partial S/\partial p}{\partial S/\partial q}$	-
"	"	q_o	"	3.5.12	-	-	1
Dependent Driver Type	3.5.4	p	Dependent Driver Type 1	3.5.15	-	-	-
Dependent Driver Type 2	3.5.5	p	Dependent Driver Type 2	3.5.17	-	-	-
Dependent Driver Type 3	3.5.6	p	Dependent Driver Type 3	3.5.19	-	-	-
Dependent Driver Type 4	3.5.7	p	Dependent Driver Type 4	3.5.21	-	-	-

Figure 3.5.1(b) shows the sensitivity network whose solution is $\frac{\partial v}{\partial R}$.

In Figure 3.5.1(b),

$$G_1 = -.02 + .00006v^2 \qquad 3.5.32$$

We observe that at the instant t_p when

$$v = v_{peak},$$

$$\frac{\partial v}{\partial t} = 0 \text{ and therefore } \frac{\partial v}{\partial R} = \frac{\partial v_{peak}}{\partial R}.$$

Thus, simuntaneous solution of the networks of Figures 3.5.1a and b will yield the desired result.

3.6 Measures of Sensitivity.

We have so far considered models for sensitivity analysis. Sensitivity analysis constitutes an important link between the analysis and design of networks. In particular, with the advent of integrated circuit techniques, design objectives based on sensitivity have become valuable (14).

For basing design objectives on sensitivity one needs non-negative real numbers based on sensitivities, called <u>measures</u> of sensitivity. In this section, measures of sensitivity based on both time-as well as frequency-domain sensitivity coefficients are discussed.

A measure of a single coefficient:

Consider the real number s defined by :

$$s = \max_t \left| \frac{\partial u}{\partial p} \right| \qquad 3.6.1$$

where u is a variable (voltage or current) of the network and p is a parameter. This number is an upper bound on the relative change in u(t), for any time t, for a small change in p. In other words, for small changes in p, $s\delta p$ is a close approximation to the maximum change δu in u, where δp is the change in p.

We can easily define a similar quantity, a vector S, for a vector U of network variables, and a single parameter p as :

$$S = \begin{bmatrix} \max\limits_{t} \left|\dfrac{\partial u_1}{\partial p}\right| \\ \vdots \\ \max\limits_{t} \left|\dfrac{\partial u_n}{\partial p}\right| \end{bmatrix} \qquad 3.6.2$$

A weighted scalar measure carrying the information about m parameters $p_1, \ldots, p_m$, is

$$J = \sum_{i=1}^{m} w_i s_i \qquad 3.6.3$$

where w_i are the weights associated with the parameters p_i and

$$s_i = \max_{t} \left|\frac{\partial u}{\partial p_i}\right| \qquad 3.6.4$$

for a scalar variable u.

If the parameters p_i are known to vary only within $p_i \pm \Delta_i$, where

$$\Delta i << p_i,$$

then

$$\max_{\substack{\text{permissible}\\ \text{changes}}} |\delta u| \approx J$$

where $w_i = \triangle_i$. Thus, the measure J in Eq. 3.6.3 is a measure of the worst-case change in u for restricted changes in p_i.
The measure

$$K = \sum_{i=1}^{m} w_i \; s_i^2 \qquad 3.6.5$$

can similarly be given an interpretation.

If p_i have a Gaussian distribution with $\sigma_i \ll p_i$, then we have

$$\sigma_u^2 \approx \sum \sigma_i^2 \left(\frac{\partial u}{\partial p_i} \right)^2 \qquad 3.6.6$$

Hence, if $w_i = \sigma_i^2$, then we have

$$\sigma_u^2 \leq K. \qquad 3.6.7$$

In In words, K is an approximate upper bound on the variance of the response variable u, when the variances σ_i^2 of the parameters p_i are small and known.

Frequency-Domain Measures for Linear Networks :

Consider a variable of the network in the frequency domain

$$U(s) = \frac{\prod_{j=1}^{m} (s - z_j)}{\prod_{j=1}^{n} (s - r_j)} \qquad 3.6.8$$

The locations of poles and zeros, that is $r_1, \ldots, r_n$ and $z_1, \ldots, z_m$ are the important factors govering U(s). The following real quantities are meaningful measures of frequency-domain sensitivity

1 $$s_k = \left| \frac{\partial r_k}{\partial p} \right| \qquad 3.6.9$$

2 $$s'_k = \left| \frac{\partial z_k}{\partial p} \right| \qquad 3.6.10$$

3 $$S = \sum_{j=1}^{m} \left|\frac{1}{z_k}\right| \left|\frac{\partial z_k}{\partial p}\right| + \sum_{j=1}^{n} \left|\frac{1}{r_k}\right| \left|\frac{\partial r_k}{\partial p}\right| \qquad 3.6.11$$

The third measure is actually a weighted sum of the first two, with the different components getting equal weightage on the logarithmic scale of frequency.

The above measures are for a single parameter p. They can be extended to several parameters either by the worst case method or by the statistical method (See Eqs. 3.6.3 and 3.6.5).

3.7 Stability analysis by sensitivity methods.

In the Liapunov sense, the motion of a system described by

$$\overset{\circ}{X} = F(X, U)$$
$$X(0) = X_o \qquad 3.7.1$$

is called locally stable if for any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\left| X_o - X'_o \right| < \delta_\epsilon \Rightarrow \left| X(t) - X'(t) \right| < \epsilon \ \forall\ t \qquad 3.7.2$$

where X'(t) is a motion of the same system, governed by

$$\overset{\circ}{X} = F(X, U) \qquad 3.7.3$$

starting from the initial state

$$X(0) = X'_o \qquad 3.7.4$$

(see (17)).

If the coefficients

$$s_{ij} = \max_{t} \left| \frac{\partial x_i}{\partial x_{o_j}} \right| \qquad 3.7.5$$

are all finite, then for

$$\delta_{\epsilon} = \frac{\epsilon}{\sum_{i} \sum_{j} s_{ij}} \qquad 3.7.6$$

the condition 3.7.2 is satisfied. If any s_{ij} is infinite, then the condition fails for the initial state

$$X'_o = X_o + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad 3.7.7$$

where ϵ is in the jth place and is arbitrarily small but non-zero. Thus we have proved that the condition that s_{ij} are all finite is equivalent to the condition 3.7.2. For finite intervals of time, s_{ij} and their combinations indicated in the previous section serve as measures of instability of the system.

3.8 Further Remarks.

Sensitivity coefficients of an arbitrary order can be obtained for a linear network, given only that amount of information which is needed for its analysis. For a nonlinear network, however, additional information is required for each order of differentiation. A procedure is established only for first order sensitivities in the

case of nonlinear networks, for it seems neither practical nor worthwhile to ask for the large amount of information which is needed for higher order coefficients. There is, however, no theoretical difficulty in arriving at these.

It seems appropriate at this place to note that Leeds and Ugron (10) have indicated a method of finding the sensitivity of a single variable of the network to all its parameters by solving only one sensitivity network. The method is useful for only linear networks.

The question of measures of sensitivity will be further discussed in the next chapter in the light of a design problem.

4. Optimization Studies.

4.1 Introduction.

We have so far developed models for sensitivity studies, essentially another facet of analysis which extends the range of possible use of the existing methods of analysis. Our concern now is to direct these analytical studies for design purposes. In this context, optimization studies constitute an essential link between analysis and design. For example, one might be expected to select a best network from a continuous class of similar networks based on an appropriate design criterion. That is, a proper choice of parameters of the network within the specified bounds satisfying an objective criterion is an important facet of design studies.

Optimization studies, such as the problem of determining a network satisfying the condition of sensitivity minimization have received attention in the literature (1,14). The very nature of these design studies is such that they do not suggest unique methods of solution.

In the following section we shall formulate the parameter optimization problem for a linear network and a general design criterion. The formulation is in terms of the values of the components; it leads to the solution of the problem as a succession of linear programming problems. Some advantages of the use of component parameters as the basis for optimization studies are pointed out.

Another problem that is discussed is that of optimal control of a network. In other words, optimal values of the independent drivers in a given network are determined as functions of time on the

basis of an appropriate design criterion. This problem has been thoroughly discussed in control theory, but without reference to the internal structure of the system (11,18).

The solution to such a problem can be obtained in terms of a two-point boundary value problem for the state equations of the network, and another set of first-order differential equations of similar dimension derived from the state equations, which are called co-state equations of the network.

We show that the co-state equations admit a simple network interpretation, which is useful in systematizing the formulation of these equations. Besides, it may also be possible to use network analysis programmes such as the one referred to in chapter 2 for the solution of the resulting two-point boundary value problem in some cases. These possibilities are explicitly pointed out. The theoretical background required for the development of the model is discussed in a separate section.

In some cases, the solution to the problem has to be further tested for optimality. Sufficiency conditions for optimality have been discussed by Robbins (21). It is demonstrated that these conditions do not depend on the graph of the network and therefore do not admit a network interpretation.

4.2 Parameter Optimization - A Component-Level Formulation.

Classical methods of network synthesis do not take into account the effect of non-ideality of components on network behaviour. The main emphasis in synthesis procedures has been to realize a network having specified nominal performance with as few components as possible.

In large networks, differences between individual component values may have considerable effect on the response which may become unacceptable merely because of this reason. The problem is particularly serious in integrated circuits, where component replacement is not possible. On the other hand, the importance of the number of components is not great in integrated circuits.

As a consequence of such developments in the technology of network construction, considerable interest has been shown in recent work in associating sensitivity studies with network synthesis (1,8,10,14). A parallel development has also taken place in the literature on automatic control; an extensive review of such development can be found in (6).

Schoeffler (14) has emphasized the importance of sensitivity analysis in the design of networks and has presented a method of sensitivity minimization. Bolstein (1) has formally analyzed Schoeffler's approach and indicated a more general class of problems of which it is a special case. He has also suggested other criteria for optimization based on sensitivity.

In this section, we shall formulate the problem of the minimization of a general objective function. The formulation uses finite-difference equations describing the manifold in the parameter space which constitutes a set of continuously equivalent networks. The procedure uses the values of the components as the variables for formulation. This procedure is helpful in reducing the algebraic manipulation to a minimum both in the formulation as well as solution of the problem.

This section is concluded with an example to illustrate the method. Furthermore, the example also indicates that it is sometimes possible to improve the insensitivity of a network without increasing the number of components; moreover it may be possible for a network containing a smaller number of components than the general member of a continuously equivalent class to be the most insensitive one. Only linear networks are considered.

Formulation

We start from a network which realizes the desired network function and refer to it as the "starting network".

Now consider all networks which have the same configuration as the starting network, and which realize the same network function, but differ from the starting network, and from each-other, in the values of their components. Each network of this set can be represented by the vector of its component values :

$$P = \text{col.}\,(p_1, \ldots, p_n) \tag{4.2.1}$$

The collection $\underline{P}$ of all such n-vectors,

$$\underline{P} = \left\{ P \;\middle|\; T(P) = T_o \right\} \tag{4.2.2}$$

where $T(P)$ is the specified network function, and T_o its desired value, forms a **manifold** in R^n. The above set-theoretic notation is used to conform to the standard notation in the literature.

For P to be non-trivial (i.e. for it to contain more than just one point) the starting network must have some built-in redundancy. (We shall return to this point when discussing the example.)

Consider two neighbouring points, P and P + δP, on $\underline{P}$. We have, retaining only the first-order term of the Taylor's expansion for $T(P + \delta P)$:

$$T(P + \delta P) = T(p) + \left\langle \frac{\partial T}{\partial P}, \delta P \right\rangle \tag{4.2.3}$$

where

$$\frac{\partial T}{\partial P} = \text{Col.}\left(\frac{\partial T}{\partial p_1}, \frac{\partial T}{\partial p_2}, \ldots, \frac{\partial T}{\partial p_n} \right) ; \tag{4.2.4}$$

and $\langle \cdot , \cdot \rangle$ denotes the scalar product, that is

$$\langle x, y \rangle = \sum_i x_i \, y_i \tag{4.2.5}$$

$$\text{Since } T(P + \delta P) = T(P) = T_o \tag{4.2.6}$$

We have,

$$\left\langle \frac{\partial T}{\partial P}, \delta P \right\rangle = 0 \tag{4.2.7}$$

The left-hand member of Eq. 4.2.7 is a function of the Laplace variable s, and hence this equation will give rise to several constraints of the type :

$$\langle F_i(P), \delta P \rangle = 0 \; ; \; i = 1, 2, \ldots, k. \tag{4.2.8}$$

The number k of these constraints depends on the complexity of $T_o(s)$. One way of the obtaining such a set of constraints is to write

$$\left\langle \frac{\partial T}{\partial P}, \delta P \right\rangle = \frac{m_1(s)}{n_1(s)} \tag{4.2.9}$$

where $m_1(s)$ and $n_1(s)$ are polynomials in s, having no common factors, and the coefficients of $m_1(s)$ are functions of $\mathbf{p}$ and $\delta\mathbf{p}$; and then equate each power of s in $m_1(s)$ separately to zero.

Another method of obtaining a set of constraints is to write

$$T(s) = \frac{a_0 + a_1 s + \ldots\ldots + a_q s^q}{b_0 + b_1 s + \ldots\ldots + b_r s^r} \qquad 4.2.10$$

where $a_q = 1$

and then write

$$\left\langle \frac{\partial a_i}{\partial P}, \delta P \right\rangle = 0; \; i = 0, 1, \ldots\ldots, q - 1, \qquad 4.2.11$$

and

$$\left\langle \frac{\partial b_i}{\partial P}, \delta P \right\rangle = 0 \; ; \; i = 0, 1, \ldots\ldots, r \qquad 4.2.12$$

In arriving at Eq. 4.2.8 we have truncated the Taylor's expansion for $T(P + \delta P)$ after the first-order term. Hence, we must impose bounds on $|\delta p_i|$ in order to meet the accuracy requirements, typically :

$$|\delta p_i| \leq \triangle_i \quad , \; i = 1, \ldots\ldots, n \qquad 4.2.13$$

The values of $\triangle_i$ in Eq. 4.2.13 are based on a suitable criterion for the control of per-step error.

For example, we may require the design function $T(s)$ to be held within $\pm\ \epsilon$ per unit in the useful interval of frequencies

$$w_1 \leq w \leq w_2$$

on the jw axis. In this case, the following rule can be employed :

$$\Delta_i = \frac{\epsilon}{n \times \max\limits_{w_1 \leq w \leq w_2} \left\{ \left| \frac{1}{T(jw)} \cdot \frac{\partial T(jw)}{\partial p_i} \right| \right\}} \qquad 4.2.14$$

for each i = 1,2,...,n.

Alternatively, we may decide to control the error in each term of the expansion of T(S) given by Eq. 4.2.10 so that a certain accuracy is maintained at any frequency. In such a case, the following rule can be used :

$$\Delta_i = \frac{\epsilon}{n \times \max \left\{ \left| \frac{1}{a_o} \cdot \frac{\partial a_o}{\partial p_i} \right|, \ldots\ldots\ldots, \left| \frac{1}{a_{q-1}} \cdot \frac{\partial a_{q-1}}{\partial p_i} \right|, \left| \frac{1}{b_o} \cdot \frac{\partial b_o}{\partial p_i} \right|, \ldots\ldots, \left| \frac{1}{b_r} \cdot \frac{\partial b_r}{\partial p_i} \right| \right\}} \qquad 4.2.15$$

In addition, we must enforce

$$\delta p_i \geq - p_i \qquad 4.2.16$$

in order to maintain passivity.

We can combine Eqs. 4.2.13 and 4.2.16 into one simple set of constraints by means of the transformation :

$$\Delta_i'' = \min \left\{ p_i, \Delta_i \right\}, \qquad 4.2.17$$

$$\delta p'_i = \delta p_i + \Delta_i'', \qquad 4.2.18$$

and

$$\Delta_i' = \Delta_i + \Delta_i'' \qquad 4.2.19$$

In terms of the new variables $\delta p'_i$ and Δ'_i we have :

$$0 \leq \delta p'_i \leq \Delta_i \; ; \; i = 1, \ldots, n \tag{4.2.20}$$

In terms of the transformed variables Eqs. 4.2.8 are as follows :

$$\langle F_i(P), \delta P' \rangle = \langle F_i(P), \Delta'' \rangle \tag{4.2.21}$$

where

$$\delta P' = \text{Col.}\,(\delta p'_1, \ldots, \delta p'_n),$$

and

$$\Delta'' = \text{Col.}\,(\Delta''_1, \Delta''_2, \ldots, \Delta''_n).$$

Let ϕ be the measure of performance to be minimized :

$$\phi = \phi\,(P) \tag{4.2.22}$$

Then, using the steepest-decent technique we can minimize at each step the change :

$$\delta \phi = \left\langle \frac{\partial \phi}{\partial P}, \delta P \right\rangle \; ; \tag{4.2.23}$$

or equivalently :

$$\delta \phi' = \left\langle \frac{\partial \phi}{\partial P}, \delta P' \right\rangle \tag{4.2.24}$$

At some stage we shall observe

$$\min\,(\delta \phi) = 0$$

or,

$$\min\,(\delta \phi') = \left\langle \frac{\partial \phi}{\partial P}, \Delta'' \right\rangle \tag{4.2.25}$$

which ensures that a minimum has been reached.

At this point, a discussion of the merits of the above formulation seems to be appropriate. We have throughout used the values of the components arranged in the vector P as the variables for the formulation. This approach eliminates the additional algebraic manipulation involved in the transformation of the results to the values of components, which will be necessary if a transformed set of variables is used for the formulation.

An examination of Eqs. 4.2.14, 4.2.15 will demonstrate that any error criterion can be directly translated into bounds on the step size of the variation of parameters for the purposes of optimization.

The formulation of the sensitivity minimization problem given in (14) is based on the assumption that the network contains a component of each type between every pair of nodes. For example, in this treatment, in an LC network, the general member of the continuously equivalent class used for sensitivity minimization would contain an L as well as a C between every pair of nodes. The necessity for such elaborate configuration does not arise in the present formulation.

Furthermore, the trajectory on P chosen in (14) may not contain the minimum number of constraints, and thus may lead to restricting the choice to a subset of $\underline{P}$. In the present formulation, however, the constraints chosen are a necessary set of conditions for the invariance of T(s) and hence are nonrestrictive within $\underline{P}$.

Newcomb (22) has raised the question as to whether one could obtain all possible realizations of the network function

which have the same graph as the starting network by the process of choosing neighbouring points on $\underline{P}$. Since we have formulated finite-difference constraints which describe $\underline{P}$ in Eq. 4.2.7, this question boils down to the question of connectedness of $\underline{P}$. The manifold $\underline{P}$ may be simply connected, like a circle, or may consist of more than one part, like a hyperbola. If $\underline{P}$ consists of more than one part, then Eq. 4.2.7 will clearly restrict the solution to that simply connected part of $\underline{P}$ to which the starting network belongs.

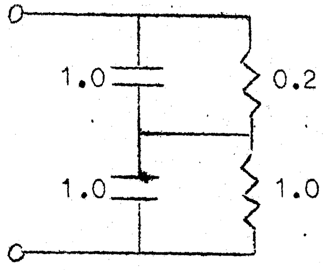
Another point that must be raised in this respect is that of the existence of local minima of the objective function. If such minima exist, the desired absolute minimum may not be obtained directly by the procedure discussed above. The study of these properties can be made only for particular types of networks and objective functions.

Example 4.2.1

Consider the design of a realization of the driving-point impedance

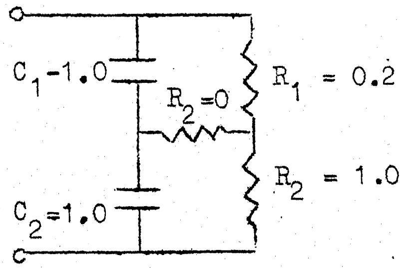
$$T_o = Z(s) = \frac{1}{s+1} + \frac{1}{s+5} \qquad 4.2.26$$

A realization of Z(s) is given in fig. 4.2.1(a). This realization does not contain any redundancy. We introduce a zero resistance R_2 to obtain a non-trivial continuously equivalent class $\underline{P}$. (See fig. 4.2.1(b)) The network in fig. 4.2.1(b) is a special case of the non-series-parallel realization described by Lee (8). In terms of the component values, or parameters, the driving-point impedance Z(s) is expressed as :



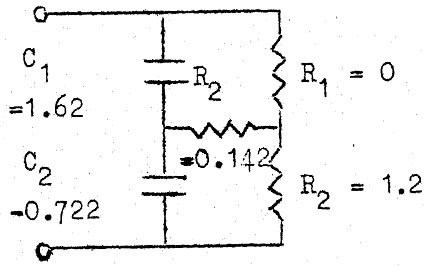
(a)

A Canonical Realization of Z(s).



(b)

The Realization in (a) with Redundancy.



(c)

The Transformed Network.

Fig. 4.2.1 Sensitivity Optimization of an R-C network.

$$Z(s) = \frac{s(p_1 + p_2)\, S_2 + (p_3 + p_5)}{s^2\, p_1\, p_2\, S_2 + s \quad p_1(p_3 + p_4) + p_2(p_4 + p_5) \quad + 1}$$

4.2.27

where

$$S_2 = p_3\, p_4 + p_4\, p_5 + p_5\, p_3 \;;$$

and

$$P = \text{Col.}\,(C_1,\, C_2,\, R_1, R_2,\, R_3).$$

Equation 4.2.8 gives rise to the following constraints :

$$S_2\, \delta p_1 + S_2\, \delta p_2 + (p_1 + p_2)\,(p_4 + p_5)\, \delta p_3$$

$$+ (p_1 + p_2)\,(p_3 + p_5)\, \delta p_4 + (p_1 + p_2)\,(p_3 + p_4)\, \delta p_5 = 0;$$

$$\delta p_3 + \delta p_5 = 0\;;$$

$$p_2\, S_2\, \delta p_1 + p_1\, S_2\, \delta p_2 + p_1\, p_2\,(p_4 + p_5)\, \delta p_3$$

$$+ p_1\, p_2 (p_3 + p_5)\, \delta p_4 + p_1\, p_2\,(p_3 + p_4)\, \delta p_5 = 0\;;$$

and

$$(p_3 + p_4)\, \delta p_1 + (p_4 + p_5)\, \delta p_2 + p_1\, \delta p_3 + (p_1 + p_2)\, \delta p_4$$

$$+ p_2\, \delta p_5 = 0.$$

Following Bolstein's (1) suggestion, let us choose

$$\phi = \sum_{i=1}^{5} \left(\frac{\partial q_1}{\partial p_i} \right)^2$$

4.2.28

where q_1 is the position of the pole at $s = -1$.

Therefore,

$$\phi = \sum_{i=1}^{5} \left(\frac{\partial N/\partial p_i}{\partial N/\partial s} \Bigg|_{s=1} \right)^2 \qquad 4.2.29$$

where N(s) is the denominator of Z(s) ;

$$N(s) = s^2 \; p_1 p_2 \; S_2 + s \quad p_1(p_3 + p_4) + p_2(p_4 + p_5) \; + 1 \qquad 4.2.30$$

The value of ϕ for the starting network is

$$\phi_o = 2.0 \; .$$

Using this criterion, the optimum shown in fig. 4.2.1(c) is obtained. The optimum value of ϕ is 1.6. It is interesting to note that the optimization has resulted in the value $R_1 = 0$, and thus the optimal network contains only as many components as the starting canonical realization.

The above example brings out an interesting conclusion. For the criterion involving pole sensitivity, it is possible that a special member of continuously equivalent class, which has fewer components than a general member of that class, can be more insensitive than a general member. Leeds and Ugron (10) have conjectured on the basis of experimental evidence that the sensitivity decreases as the number of elements increases in continuously equivalent network. This conjecture is clearly not applicable to continuously equivalent networks in general, as demonstrated by the above example.

4.3 Optimal Control and Pontryagin's Principle.

In this section, we shall discuss two problems of optimal control; the terminal control problem and the minimum integral problem. These problems can be transformed into two-point boundary value problems by a principle based on variational calculus, called the Pontryagin's Maximum Principle. The Principle and its application to the terminal control problem will be discussed. A method of converting the minimum-integral problem into a terminal-control problem will be discussed to show the generality of the latter problem. A detailed discussion of this material can be found in (18).

Terminal Control Problem

Given: a system described by the equations

$$\overset{o}{X} = F(X, U) \tag{4.3.1}$$

where X is the state-vector of dimension n and U is the input or control vector of dimension m;

the initial state X_o;

a final time T ;

and a constant vector B of dimension n; to minimize

$$J = B' X(T) \tag{4.3.2}$$

(where prime denotes transpose)

by choosing the appropriate control

$$u^o = u^o(t) \ ; \quad 0 \leq t < T \tag{1.5.3}$$

Pontryagin's Maximum Principle :

The optimal input $u^o(t)$ minimizing the criterion of Eq. 4.3.2 also maximizes the Hamiltonian $H(t)$ given by

$$H(t) = P'(t)\, F(X(t); U(t)) \tag{4.3.4}$$

at each instant of time $0 \leq t \leq T$; where $P(t)$ is the solution to the differential equations

$$\frac{d\, p_i}{dt} = \frac{\partial H(t)}{\partial x_i} ,$$

which can be written for convenience as

$$\frac{dP}{dt} = \frac{\partial H}{\partial X} \; ; \tag{4.3.5}$$

with the terminal conditions

$$P(T) = -B \tag{4.3.6}$$

Thus, the problem of searching for a control input function is reduced to solving a two-point boundary-value problem consisting of Eqs. 4.3.1, 4.3.6.

Minimum-Integral Problem

Given the system and the conditions of the first problem, to minimize

$$J' = \int_0^T f_{n+1}\,(X, U)\, dt \tag{4.3.7}$$

This problem can be converted into the first problem as follows :

Augment the system equations by the equation:

$$\frac{d\,x_{n+1}}{dt} = f_{n+1}(X,U)$$

with the initial conditions

$$x_{n+1}(0) = 0$$

Clearly,

$$x_{n+1}(t) = J' \ .$$

Thus, if we let

$$b = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ (n \text{ times}) \\ 1 \end{bmatrix}$$

then for the augmented state equations we have the first problem. Component-level procedures for handling the terminal control problem will be discussed in the following sections. The minimum-integral problem has been discussed here to show an example where the first problem covers a frequently used objective function. More such examples can be found in (11, 18).

4.4 Optimal Inputs in the Time-domain

Problems of the choice of optimal inputs to dynamical systems arise in every field of engineering, from the operation of a chemical plant to the directing of a missile.

The task of finding the optimal input function, usually from a continuum of permissible inputs, is reduced by Pontryagin's maximum principle to that of solving a 2-point boundary-value

problem. The maximum principle is especially significant for linear systems, since in this case it constitutes a necessary and sufficient condition for optimality.

In the case of linear systems, several advantages of retaining the structural details of the system in the analysis have been shown (6). Among them are the freedom of interconnections of subsystems and a physical meaning for each system variable.

We shall outline a procedure for forming the state equations for an RLC Network, and show that the corresponding co-state equations belong to a similar network. While electrical terminology is used for convenience, the analysis is applicable to similar non-electrical systems.

We recall from the previous section that the co-state equations for a system satisfying the state equations :

$$\overset{\circ}{X} = F(X,U,\ t) \tag{4.4.1}$$

are

$$\overset{\circ}{P} = -\frac{\partial H}{\partial X} \tag{4.4.2}$$

where

$$H = P'\ F \tag{4.4.3}$$

Linear Networks: From the analysis of Chapter 2, we have the state equations for linear networks :

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_5 \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \cdot \right.$$

$$\times \left\{ U + \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \right\}^{-1} \left. \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} 0 & -A_{32} \\ -B_{12} & 0 \end{bmatrix} \right\} \begin{bmatrix} V_2 \\ I_5 \end{bmatrix}$$

$$+ \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{23} \\ -B_{21} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \left\{ U + \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \right\}^{-1} \right.$$

$$\left. \times \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} 0 & -A_{33} \\ -B_{11} & 0 \end{bmatrix} \right\} \begin{bmatrix} V_1 \\ I_6 \end{bmatrix} \qquad 4.4.4$$

The co-state equations of a linear system

$$\overset{o}{X} = P_1 X - QU \qquad 4.4.5$$

for linear criteria are

$$\overset{o}{P} = - P_1' P$$

which in this case become

$$\frac{d}{dt}\begin{bmatrix} P_2 \\ P_5 \end{bmatrix} =$$

$$\left\{ \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \left\{ U + \begin{bmatrix} -R_3 & 0 \\ 0 & -G_4 \end{bmatrix} \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \right\}^{-1} \begin{bmatrix} -R_3 & 0 \\ 0 & -G_4 \end{bmatrix} \begin{bmatrix} 0 & -A_{32} \\ -B_{12} & 0 \end{bmatrix} \right.$$

$$\times \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} P_2 \\ P_5 \end{bmatrix} \qquad 4.4.6$$

Substituting the transformation

$$\begin{bmatrix} P_2 \\ \\ P_5 \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} V^*_2 \\ \\ I^*_5 \end{bmatrix} \qquad 4.4.7$$

into Eq. 4.3.6 we get

$$\frac{d}{dt} \begin{bmatrix} V^*_2 \\ \\ I^*_5 \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & -A_{22} \\ \\ -B_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ \\ -B_{23} & 0 \end{bmatrix} \left\{ U + \begin{bmatrix} -R_3 & 0 \\ \\ 0 & -G_4 \end{bmatrix} \right.\right.$$

$$\left.\left. x \begin{bmatrix} 0 & A_{31} \\ \\ B_{13} & 0 \end{bmatrix} \right\}^{-1} \begin{bmatrix} -R_3 & 0 \\ \\ 0 & -G_4 \end{bmatrix} \begin{bmatrix} 0 & -A_{32} \\ \\ -B_{12} & 0 \end{bmatrix} \right\} \begin{bmatrix} V^*_2 \\ \\ I^*_5 \end{bmatrix} \qquad 4.4.8$$

Equations 4.4.8 resemble Eq. 4.4.4. The networks corresponding to them are identical except for two differences :

(i) All resistance values are made negative

(ii) Voltage and current driver terms are set to zero.

We shall call the network corresponding to Eq. 4.4.8 the adjoint network. It also follows from transformation 4.4.7 that the co-state vector P consists of charge and flux variables of the adjoint network. On the other hand, if the state variables of theoriginal network are chosen to be charge and flux variables, then the corresponding co-state vector consists of the voltage and current variables of the adjoint network.

Multiterminal Components

It is best suited to this analysis to treat multiterminal components as they are, that is, to use Eq. 2.10.3 for their

terminal equations. In that case the matrix

$$\begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix}$$

appearing in the above analysis ceases to be diagonal. It can be easily shown that the corresponding matrix in the co-state equations is the negative transpose of the original matrix. Thus, for example, if a transistor with the terminal equations

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \qquad 4.4.9$$

is present in the original network, its counterpart in the adjoint has the terminal equations

$$\begin{bmatrix} v^*_1 \\ i^*_2 \end{bmatrix} = \begin{bmatrix} -h_{11} & -h_{21} \\ -h_{12} & -h_{22} \end{bmatrix} \begin{bmatrix} i^*_1 \\ v^*_2 \end{bmatrix} \qquad 4.4.10$$

This result when translated into the language of dependent drivers reads as follows.

Replace a dependent driver of the original network with one in which the roles of the controlling and controlled edges are interchanged, and the parameter is made negative.

The type of the dependent driver. changes according to table 4.4.1

Table 4.4.1 : Types of dependent drivers in original and co-state networks.

Old Type	New Type
1	4
2	2
3	3
4	1

Solution

In general the optimal control problem requires the solution of a two-point boundary value problem, because if for the system equations the initial conditions are specified, for the adjoint equations the final conditions are specified. In the above case, however, the co-state equations are not coupled with the state equations, as can be seen from the fact that the adjoint network is not coupled to the original network. Hence, the co-state equations can be integrated backwards in time, from T to 0, starting from the value of P(T) given by Eq. 4.3.6 . Then, using the solution $P(t)$, the optimal inputs U^{o} which maximize H can be found and used in the state equations yielding the optimal solution of the system.

It may be noted here that integrating backwards in time is equivalent to integrating forward after having changed the signs of the dynamic components of the adjoint network. That is, the signs of components L and C are opposite to their signs in the original network.

Other Criteria

It was shown in Section 4.3 that these problems in which an integral criterion is to be minimized can be converted to the generalized mode by suitably augmenting the system equations. The co-state equations for the augmented set :

$$\overset{o}{X} = P X + QU$$

$$\overset{o}{X}_{n+1} = f_{n+1}(X, U, t) \qquad 4.4.11$$

are

$$\overset{o}{P} = - P'P + \frac{\partial f_{n+1}}{\partial X}$$

$$\overset{o}{P}_{n+1} = 0 \qquad 4.4.12$$

Here, the notation

$$\frac{\partial f_{n+1}}{\partial X}$$

is used to represent the vector

$$\begin{bmatrix} \frac{\partial f_{n+1}}{\partial x_1} \\ \vdots \\ \frac{\partial f_{n+1}}{\partial x_n} \end{bmatrix}$$

in a concise form. Such a notation in future should be interpreted in this manner.

In Eq. 4.4.12, P is still an n-dimensional vector.

The network corresponding to the n-dimensional part of Eq. 4.3.12 can be generated from the adjoint network described earlier by connecting a current driver of value

$$-\frac{\partial f_{n+1}}{\partial v_{C_i}} \qquad 4.4.13$$

in parallel with each capacitor C_i and a voltage driver of value

$$-\frac{\partial f_{n+1}}{\partial i_{L_i}} \qquad 4.4.14$$

in series with each inductor L_i. The orientation of the drivers are shown in fig. 4.4.1.

Example 4.4.1

In the network of fig. 4.4.2(a) find the waveform of the driving voltage u(t) which will produce the maximum possible voltage across the capacitor C in 10 seconds, given

$$|u(t)| \leq 1 \text{ for all } t.$$

The state equations of the above network are

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1.2 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad 4.4.15$$

The adjoint network is shown in fig. 4.3.2(b) and its equations in terms of q^*_C and ϕ^*_L are

$$\begin{bmatrix} q^*_C \\ \phi^*_L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1.2 \end{bmatrix} \begin{bmatrix} q^*_C \\ \phi^*_L \end{bmatrix} \qquad 4.4.16$$

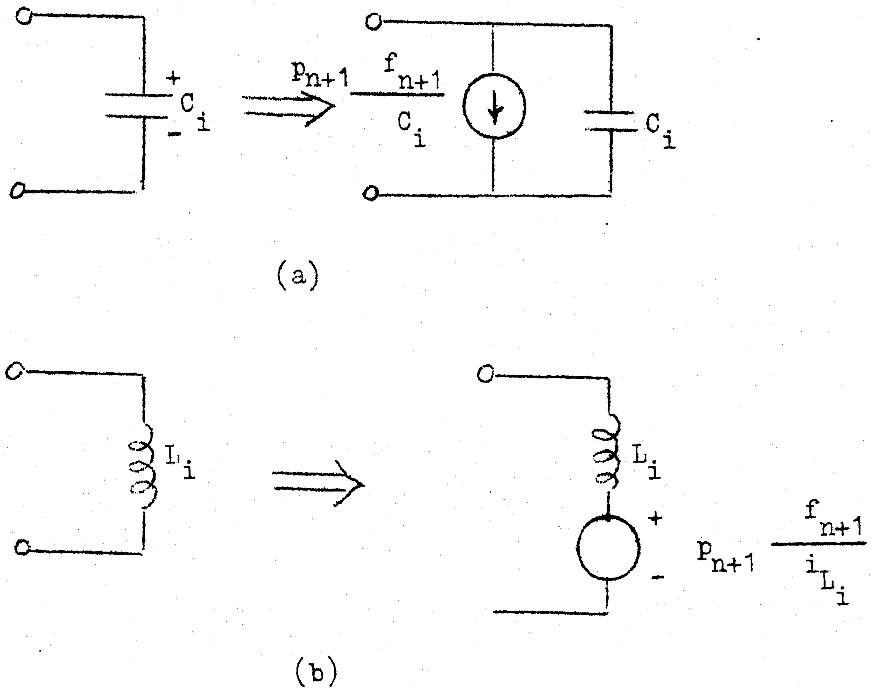
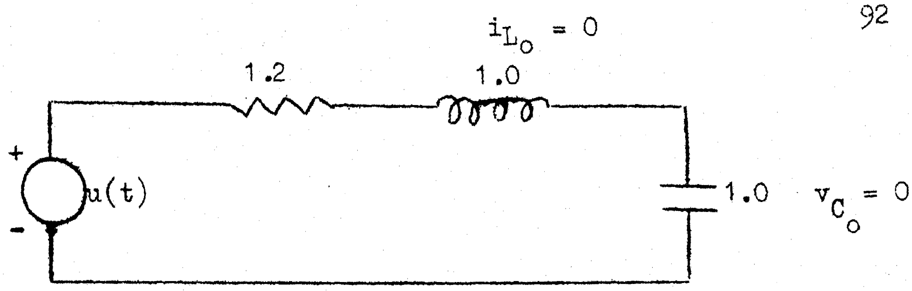
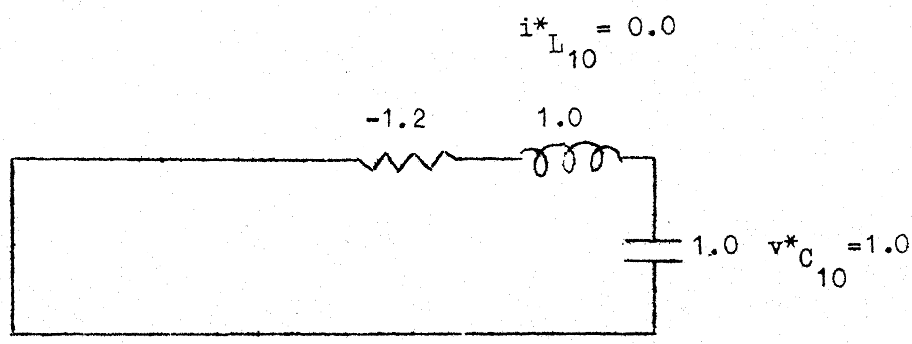


Fig. 4.4.1 Drivers in the adjoint network.



(a)

The Original network.



(b)

The adjoint network.

Fig. 4.4.2 A simple Optimal design problem.

We are to minimize $- v_C(10)$

$$\text{or} \quad \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} (10) \qquad 4.4.17$$

Therefore

$$b = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad 4.4.18$$

and hence

$$\begin{bmatrix} q^*_C \\ \phi^*_L \end{bmatrix} (10) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad 4.4.19$$

Solving Eqs. 4.4.16 and 4.4.19 we get

$$\begin{bmatrix} q^*_C \\ \phi^*_L \end{bmatrix} = 1.25 \, e^{.6(t - 10)} \begin{bmatrix} \sin (\tan^{-1} 4/2 + 8 - .8t) \\ \sin (8 - .8t) \end{bmatrix} \qquad 4.4.20$$

$u^o(t)$ is chosen to maximize

$$H = \phi^*_L \, u(t)$$

from which we get

$$u^o(t) = \mathrm{sgn}^{1} \, (\sin (8 - .8t)) \qquad 4.4.21$$

Using this value, v_C (10) (max) is obtained to be 1.2 volts.

---

1 the function signum (sgn) is defined by

$$\begin{aligned} \mathrm{sgn}\,(x) &= -1 \quad \text{if } x \quad 0 \\ &= 0 \quad \text{if } x = 0 \\ \text{and} \quad &= 1 \quad \text{if } x \quad 0 \end{aligned}$$

Nonlinear Networks :

We restate the nonlinear component terminal equations from Chapter 2 here for convenience. The mixed form of the resistor equations is used for convenience. We have :

$$\frac{d}{dt}\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} I_2 \\ V_5 \end{bmatrix}$$

and

$$\begin{bmatrix} V_2 \\ I_5 \end{bmatrix} = D_d\left(\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}\right) \tag{4.4.22}$$

for the dynamic components; and

$$\begin{bmatrix} V_3 \\ I_4 \end{bmatrix} = D_r\left(\begin{bmatrix} I_3 \\ V_4 \end{bmatrix}\right) \tag{4.4.23}$$

for the resistors. If multiterminal resistors are present, their terminal equations are included in Eq. 4.4.23. First, we shall consider networks of two-terminal components. From Pontryagin's principle we have the co-state equations :

$$\frac{d}{dt}\begin{bmatrix} P_2 \\ P_5 \end{bmatrix} = -\frac{\partial}{\partial\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}}\left\{\begin{bmatrix} P'_2 & P'_5 \end{bmatrix}\begin{bmatrix} \dot{Q}_2 \\ \dot{\phi}_5 \end{bmatrix}\right\}$$

$$- p_{n+1}\frac{\partial f_{n+1}}{\partial\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}} \tag{4.4.24}$$

or,

$$\frac{d}{dt}\begin{bmatrix} P_2 \\ P_5 \end{bmatrix} = M \begin{bmatrix} P_2 \\ P_5 \end{bmatrix} - p_{n+1} \frac{\partial f_{n+1}}{\partial \begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}} \text{, (say),} \qquad 4.4.25$$

Then

$$M = - \begin{bmatrix} \dfrac{\partial \mathring{\phi}'_2}{\partial Q_2} & \dfrac{\partial \mathring{\phi}'_5}{\partial Q_2} \\ \dfrac{\partial \mathring{Q}'_2}{\partial \phi_5} & \dfrac{\partial \mathring{\phi}'_5}{\partial \phi_5} \end{bmatrix}$$

$$= - \begin{bmatrix} \dfrac{\partial I'_2}{\partial Q_2} & \dfrac{\partial V'_5}{\partial Q_2} \\ \dfrac{\partial I'_2}{\partial \phi_5} & \dfrac{\partial V'_5}{\partial \phi_5} \end{bmatrix}$$

$$= \frac{\partial}{\partial \begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}} \left\{ \begin{bmatrix} V'_2 & I'_5 \end{bmatrix} \begin{bmatrix} 0 & A_{22} \\ B_{22} & 0 \end{bmatrix} + \begin{bmatrix} V'_3 & I'_4 \end{bmatrix} \begin{bmatrix} 0 & A_{32} \\ B_{12} & 0 \end{bmatrix} \right\} \qquad 4.4.26$$

Since

$$\frac{\partial \begin{bmatrix} V'_1 & I'_6 \end{bmatrix}}{\partial \begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}} = 0 \qquad 4.4.27$$

Differentiating Eqs. 4.4.22, 4.4.23 partially with respect to $\begin{bmatrix} Q_2 \\ \phi_5 \end{bmatrix}$

and substituting into Eq. 4.4.26 we get

$$M = D_d^{s'} \left\{ \begin{bmatrix} 0 & -A_{22} \\ -B_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -A_{21} \\ -B_{23} & 0 \end{bmatrix} \right\} U + (-D_r^{s'}) \begin{bmatrix} 0 & A_{31} \\ B_{13} & 0 \end{bmatrix} \Bigg\}^{-1}$$

$$(-D_r^{s'}) \begin{bmatrix} 0 & -A_{32} \\ -B_{12} & 0 \end{bmatrix} \Bigg\} \quad ; \qquad 4.4.28$$

where the meaning of D_d^s and D_r^s has been explained in chapter 2.

We observe that Eq. 4.4.26 is similar to Eq. 4.4.8 where the small-signal matrices D_d^s and D_r^s replace the linear component-value matrices

$$\begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} R_3 & 0 \\ 0 & G_4 \end{bmatrix}$$

Therefore, the steps for transforming the original network into the co-state network remain the same. If multiterminal components are included, the discussion in the linear case for their replacements in the co-state network holds true here also, when small-signal characteristics are considered. Therefore, we can give the following algorithm for converting the original network into the co-state network :

Algorithm 4.1.1.

(i) Replace all independent voltage drivers by short-circuits and all independent current drivers by open-circuits.

(ii) Replace all two-terminal resistors by linear resistors whose values are negative of the small-signal values of the original resistors

(iii) Reverse the roles of controlling and controlled branches for all dependent drivers and replace them by linear dependent drivers whose value is the negative of the small-signal value of original

driver and whose type is dictated by Table 4.4.1.

(iv) Replace each capacitor by a linear capacitor whose value is equal to the small-signal value of the original capacitor, and connect in parallel with it a current driver of value

$$C_i \; p_{n+1} \frac{\partial f_{n+1}}{\partial q_{C_i}} .$$

where C_i is the small-signal value of the capacitor.

Similarly, replace each inductor by a linear inductor whose value is equal to the small-signal value of the original inductor, and connect in series with it a voltage driver of value

$$L_i \; p_{n+1} \frac{\partial f_{n+1}}{\partial \phi_{L_i}}$$

where L_i is the small-signal value of the inductor :

4.5 Tests of Optimality :

In the last section the network corresponding to co-state equations was derived from the original network. In the case of nonlinear networks the adjoint network depended on the response of the original network, and hence the solution of the two was essentially coupled. In the linear case, if the criterion for optimality was also linear in the final state, then the two networks are uncoupled and the optimal control problem can be solved as an initial value problem.

If the problem is known to have a solution, and the equations have a unique solution, the problem is completely solved since the obtained solution can only be the optimal one. If either

of these conditions is unfulfilled, however, there still remains the problem of determining whether a given solution is optimal or not. Let us re-write the state and co-state equations of the linear case, for completeness :

$$\overset{o}{X} = P X + Q U \qquad 4.5.1(a)$$

$$\overset{o}{X}_{n+1} = f_{n+1}(X, U, t) \qquad 4.5.1(b)$$

$$\overset{o}{P} = -P' P - p_{n+1} \frac{\partial f_{n+1}}{\partial X} \qquad 4.5.2$$

We shall discuss some sufficiency conditions for optimality for the linear case in order to investigate their possible relation to the topology of the network. The tests discussed below are due to Robbins (21).

Sufficient conditions for optimality are available for the case where the control vector U^o to be tested is in the interior of the permissible region U. These conditions can, however, be applied to a wider class of situations by 'freezing' some of the control variables or their combinations (See (21)).

In the following analysis, we assume that the control vector is in the interior of its region.

The first condition is that the matrix

$$H_{uu} = \left[\frac{\partial^2 H}{\partial u_i \partial u_j} \right] \qquad 4.5.3$$

should be negative-definite at

$$U = U^o$$

for U^o to be optimal.

In the system described by Eq. 4.5.1 this corresponds to the condition that the matrix

$$\left[\frac{\partial^2 f_{n+1}}{\partial u_i \partial u_j} \right] \qquad 4.5.4$$

be positive definite. If the matrix 4.5.4 is positive non-definite, the solution is non-optimal. If it is positive semi-definite, further tests must be carried out.

A further test consists in reformulating the problem with possibly fewer control variables so that, for the new problem,

$$H_{uu} = \underline{0} \qquad 4.5.5$$

(see (18)). This condition may also arise directly in the original problem if the criterion f_{n+1} does not depend explicitly on U, that is

$$f_{n+1} = f_{n+1} (X, t) \qquad 4.5.6$$

In the case where Eq. 4.4.5 holds, the following two conditions are sufficient for optimality :

$$\text{(i)} \quad \frac{\partial}{\partial U} \left(\frac{d}{dt} \left(\frac{\partial H}{\partial U} \right) \right) = \underline{0} \qquad 4.5.7$$

and

$$\text{(ii)} \quad \frac{\partial}{\partial U} \left(\frac{d^2}{dt^2} \left(\frac{\partial H}{\partial U} \right) \right) . \qquad 4.5.8$$

is positive definite.

The condition of Eq. 4.5.7 is automatically satisfied by the system described by Eq. 4.5.1, and the condition of Eq. 4.5.8 reduces to the condition that the matrix

$$\left[\frac{\partial^2 f_{n+1}}{\partial x_i \partial x_j} \right] \qquad 4.5.9$$

be positive definite.

If the matrix is positive non-definite the solution is non-optimal. If the matrix is positive semi-definite, this test is also indecisive, for instance when not all the state variables influence f_{n+1} :

$$f_{n+1} = f_{n+1} (x_1, \ldots, x_k ; t) ; \; k \quad n \qquad 4.5.10$$

Further tests can be applied to this case; but we shall not go into those details, because they are not necessary for the development of this thesis.

In the above discussion we observe that both results depend only on the cost function, and not explicitly on the topology of the network. Thus, it is useless to attempt to give a network interpretation for these tests.

4.6 Further Remarks :

We have discussed some aspects of design which indicate computer application. In particular, parameter optimization can be implemented on a digital computer. Some discussion of this topic can be found in (12).

The advantages of a network interpretation of co-state equations are many. Apart from all usefulness of the formulation as an analysis tool, the procedure has a strong pedagogic advantage; it makes it possible to explain important results of optimal control theory through network analysis.

The simplicity of formulation, and its systematic nature, makes both analytical and mechanical formulations of co-state equations easy and less prove to errors.

The two-point boundary value problem in which this formulation results can in some restricted cases be solved directly by network analysis programmes, as pointed out in section 4. In some more general cases, it may be possible to solve it iteratively as a succession of initial value problems.

5. Conclusions :

The technology of network construction has developed in all facets. The rapid reduction in component sizes and costs have made extremely sophisticated electronic equipment such as rocket guidance systems feasible. Integrated circuits can be considered as the pinnacle of progress in this direction. Their advent caused an immediate change in the philosophy and the means of network designers. The need for economy in the number of components is replaced by the need for reliability and the need for design and analysis of extremely large networks. The requirement of speed in analysis has been so great that time-sharing consoles with expensive light-pen input and graphical display units are made available at several places (See (3)).

As design methods better and better suited to digital computers are developed, the cost of analysis in terms of computer equipment and time is likely to decrease. The present-day design oriented analysis programmes lay emphasis on user interaction. If some features of design are made automatic, the need for such expensive interaction will decline. As mentioned in the introduction the models that have been presented in chapters 3 and 4 provided the theoretical basis for developing programmes which would enable to minimize human interaction with the computer in the design process.

The formulation of the methods in those chapters has always been kept at the component level. Some advantages of the same in reference to parameter optimization were pointed out in the

conclusion of chapter 4. These advantages, however, have a more or less general nature.

Firstly, computation is systematized by this procedure, since once the transformed network is obtained analysis takes the same course as usual network analysis.

The other advantages are all derived from the fact that component-level formulation makes possible a physical interpretation of the quantities on which computations are made. Thus, interpretation of results and error analysis become simpler, and an insight into the role of each component in the problem can be obtained simply by studying the results.

It can be seen that while the procedures for sensitivity and co-state formulations are quite elegant for linear networks, they seem artificial for nonlinear networks, in that they require additional information about the network which is not needed for simple analysis. The situation here resembles the attempts to modify transform methods in order to make them applicable to non-linear systems. These methods had gained considerable momentum before time-domain analysis became well-known. Likewise, the studies on sensitivity and optimization for nonlinear systems will be more natural when new methods are developed than in the present studies.

BIBLIOGRAPHY

1. M.L. Bolstein "Generation of Minimum Sensitivity Networks" IEEE Trans. Cir. Th. Vol. 14, No. 1, March 1967.

2. L. Hellerman "A Computer Application to Reliable Circuit Design "IRE Trans. Reliability and Quality Control, Vol.11, No. 1, 1962.

3. J. Katzenelson "An Algorithm for Solving Nonlinear Resistor Networks" Bell System Technical Journal, Vol. 44, No. 8, Oct. 1965.

4. J. Katzenelson, L.H. Seitelman "An Iterative Method for Solution of Nonlinear Resistor Networks" Allerton Conference on Circuits and Systems, Oct. 1965.

5. H.E. Koenig, Y.Tokad, H.K. Kesavan : Analysis of Discrete Physical Systems (book) McGraw-Hill 1967.

6. P. Kokotovic', R.S. Rutman "Sensitivity of Automatic Control Systems (review)" Auto. Remote Control Vol. 26, No. 4, Apr 1965.

7. F.F.Kuo "Network Analysis by Digital Computer" Proc. IEEE Vol. 54, No. 6, June 1966.

8. Samuel Chin-Hsun Lee "Network Synthesis to Minimize multi-parameter Sensitivity" Report R-262, Co-ordinated Science Lab., University of Illinois, Aug. 1965.

9. J.V. Leeds Jr. "Transient and Steady State Sensitivity Analysis" IEEE Trans. Cir. Th. Vol. 13, No. 3, Sept. 1966.

10. J.V. Leeds Jr., G. Ugron "Simplified Multiple Parameter Sensitivity Calculations and Continuously Equivalent Networks" (Report) Series EE - 66, No. 3, Rice University June 1966.

11. L.S. Pontryagin et al : The Mathematical Theory of Optimal Processes (book) Interscience 1962.

12. R.A. Rohrer " Fully Automated Network Design by Digital Computer : Preliminary Considerations" Proc. IEEE Vol. 55, No. 11, Nov. 1967.

13. H.V. Sahasrabuddhe "General Purpose Digital Computer Programme for the Solution of Nonlinear Lumped Parameter Systems " (M. Tech. Thesis) I.I.T. Kanpur 1966.

14. J.D. Schoeffler "The Synthesis of Minimum Sensitivity Networks" IEEE Trans. Cir. Th. Vol. 11, No. 2, June 1964.

15. S.Seshu, M.B. Reed : Linear Graphs and Electrical Networks (book) Addison, Wesley 1961.

16. Special Issue on Computer-Aided Design, Proc. IEEE, Vol. 55, No. 11, Nov. 1967.

17. R.Tomovic': Sensitivity Analysis of Dynamic Systems (book) McGraw-Hill 1963.

18. J.T.Tou : Modern Control Theory (book) McGraw-Hill 1964.

19. V.F.Yevseyev, T.D. Korablina, S.I.Rozhkov "An Algorithm for Optimizing Systems With Fixed Random Parameters" Engineering Cybernatics Nov - Dec 1965.

20. H.K. Kesavan, I.G. Sharma, U.R. Prasad : "Sensitivity-state Models for Linear Systems". International J.Control (to be published).

21. H.M. Robins "A Generalized Legendre-Clebsch Condition for the Singular cases of Optimal Control" IBMJ Res. Dev. Vol. 11, No.4, July 1967.

22. R.W.Newcomb "The Noncompleteness of Continuously Equivalent Networks" IEEE Trans.Circuit Theory Vol. 13, No. 2, June 1966.

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